Module III: Relativistic ED: applications

Lecture 21: Radiation from an accelerating charge

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Oct 30, 2018

Outline

1. From Lienard-Wiechert potentials to EM fields
2. Calculating relevant derivatives, $(\partial/\partial t)|_{\vec{\mathbf{x}}}$ and $(\partial/\partial x^{\alpha})|_t$
3. Calculating $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ including their radiative components

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Motivation and notation

- We saw in the last lecture that a charge moving at constant velocity (in vacuum) cannot radiate. Now is the time to check what happens when the charge is accelerating.
- Let the motion of the charge q be described by $\vec{\mathbf{x}}'(t_r)$.
- The velocity of the charge is then $\vec{\mathbf{v}}(t_r) = d\vec{\mathbf{x}}'(t_r)/dt_r$ and its acceleration is $\vec{\mathbf{a}}(t_r) = d\vec{\mathbf{v}}(t_r)/dt_r$.
- We shall denote $\vec{\mathbf{r}} = \vec{\mathbf{x}}(t) - \vec{\mathbf{x}}'(t_r)$, and $r = |\vec{\mathbf{r}}|$. Both of these are in principle functions of $\vec{\mathbf{x}}$, t and the trajectory $\vec{\mathbf{x}}'(t_r)$.
- In order to find the potentials and fields at $(\vec{\mathbf{x}}, t)$, the motion of the source at the retarded time t_r is relevant. The location of the charge at the retarded time is $\vec{\mathbf{x}}'(t_r)$, its velocity $\vec{\mathbf{v}}(t_r)$, and its acceleration $\vec{\mathbf{a}}(t_r)$.
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The need for change of variables

- The Lienard-Wiechert potentials are

$$\phi(\vec{\mathbf{x}}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{s(\vec{\mathbf{x}}, t, \vec{\mathbf{x}}'(t_r))} , \quad \vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\vec{\mathbf{v}}(t_r)}{s(\vec{\mathbf{x}}, t, \vec{\mathbf{x}}'(t_r))} \qquad (1)$$

Note: $\vec{\mathbf{x}}'(t_r) = \vec{\mathbf{x}}'[t_r(x, t)]$

- For obtaining the EM fields, we need to calculate

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = -\nabla\phi(\vec{\mathbf{x}}, t) - \frac{\partial \vec{\mathbf{A}}(\vec{\mathbf{x}}, t)}{\partial t} , \quad \vec{\mathbf{B}}(\vec{\mathbf{x}}, t) = \nabla \times \vec{\mathbf{A}}(\vec{\mathbf{x}}, t) \qquad (2)$$

where ∇ and $\partial/\partial t$ are the partial derivatives taken by keeping t and $\vec{\mathbf{x}}$, respectively, constant. I.e.

$$\nabla_\alpha = \left.\frac{\partial}{\partial x^\alpha}\right|_t , \quad \frac{\partial}{\partial t} = \left.\frac{\partial}{\partial t}\right|_{\vec{\mathbf{x}}} \qquad (3)$$

- Since the expressions for ϕ and A above are in terms of t_r, which further depends on $\vec{\mathbf{x}}$ and t, care needs to be taken while taking the derivatives.

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Two ways of determining $(\partial r/\partial t)|_{\vec{\mathbf{x}}}$

- The quantity $r = |\vec{\mathbf{x}} - \vec{\mathbf{x}}'(t_r)|$ may be written in multiple ways:
  - $r(x,t) = |\vec{\mathbf{x}} - \vec{\mathbf{x}}'[t_r(x,t)]|$ gives

$$\left.\frac{\partial r}{\partial t}\right|_{\vec{\mathbf{x}}} = \frac{\partial r}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial t_r} \left.\frac{\partial t_r}{\partial t}\right|_{\vec{\mathbf{x}}} = -\frac{\vec{\mathbf{r}}}{r} \cdot \vec{\mathbf{v}} \left.\frac{\partial t_r}{\partial t}\right|_{\vec{\mathbf{x}}} . \quad (4)$$

  - Finally, $r(\vec{\mathbf{x}},t) = c \times [t - t_r(\vec{\mathbf{x}},t)]$ gives

$$\left.\frac{\partial r}{\partial t}\right|_{\vec{\mathbf{x}}} = c - c \left.\frac{\partial t_r}{\partial t}\right|_{\vec{\mathbf{x}}} \quad (5)$$

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Calculating $(\partial/\partial t)|_{\vec{\mathbf{x}}}$ from $(\partial/\partial t_r)|_{\vec{\mathbf{x}}}$

- From the last two equations, we get

$$c - c\left.\frac{\partial t_r}{\partial t}\right|_{\vec{\mathbf{x}}} = -\frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{r}\left.\frac{\partial t_r}{\partial t}\right|_{\vec{\mathbf{x}}} \tag{6}$$

That is,

$$\left.\frac{\partial t_r}{\partial t}\right|_{\vec{\mathbf{x}}} = \frac{1}{1-\frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{rc}} = \frac{r}{s} \tag{7}$$

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an important step in getting derivatives in terms of $\vec{\mathbf{x}}$ and t.

- Now that we know how to calculate $(\partial/\partial t)|_{\vec{\mathbf{x}}}$, we'll go on to calculate $(\partial/\partial x^\alpha)|_t$.

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- Also, $r(\vec{\mathbf{x}},t) = c \times [t - t_r(\vec{\mathbf{x}},t)]$, one gets

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- This gives

$$\left.\frac{\partial t_r}{\partial x^\alpha}\right|_t = -\frac{\vec{\mathbf{r}}/r}{c - \frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{r}} = -\frac{\vec{\mathbf{r}}}{sc} \tag{12}$$

- Thus,

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- The space derivative $\nabla = (\partial/\partial x^\alpha)|_t$ is then

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- This gives

$$\left.\frac{\partial t_r}{\partial x^\alpha}\right|_t = -\frac{\vec{\mathbf{r}}/r}{c - \frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{r}} = -\frac{\vec{\mathbf{r}}}{sc} \tag{12}$$

- Thus,

$$\left.\frac{\partial}{\partial x^\alpha}\right|_t = \left.\frac{\partial}{\partial x^\alpha}\right|_{t,t_r} - \frac{\vec{\mathbf{r}}}{sc} \left.\frac{\partial}{\partial t_r}\right|_t . \tag{13}$$

- The space derivative $\nabla = (\partial/\partial x^\alpha)|_t$ is then

$$\nabla\Big|_t = \nabla\Big|_{t,t_r} - \frac{\vec{\mathbf{r}}}{sc} \left.\frac{\partial}{\partial t_r}\right|_t \tag{14}$$

Outline

Calculating $\vec{\mathbf{E}}$

- Now we calculate

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = \frac{q}{4\pi\epsilon_0}\left[-\nabla\frac{1}{s(t_r)} - \frac{1}{c^2}\frac{\partial}{\partial t}\frac{\vec{\mathbf{v}}(t_r)}{s(t_r)}\right] \qquad (15)$$

or

$$\frac{4\pi\epsilon_0}{q}\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = \frac{1}{s^2}\nabla s - \frac{1}{c^2}\frac{\partial}{\partial t}\frac{\vec{\mathbf{v}}(t_r)}{s(t_r)} \ . \qquad (16)$$

- Using the expressions for ∇ and $(\partial/\partial t)$ obtained earlier in this lecture,

$$\frac{4\pi\epsilon_0}{q}\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = \frac{1}{s^2}\nabla s\bigg|_{t,t_r} - \frac{\vec{\mathbf{r}}}{s^3 c}\frac{\partial s}{\partial t_r}\bigg|_{t} + \frac{1}{c^2}\frac{r}{s}\frac{\vec{\mathbf{v}}}{s^2}\frac{\partial s}{\partial t_r}\bigg|_{\vec{\mathbf{x}}} - \frac{1}{c^2}\frac{r}{s}\frac{1}{s}\vec{\mathbf{a}}\bigg|_{\vec{\mathbf{x}}} \qquad (17)$$

where $\vec{\mathbf{a}}(t_r) = d\vec{\mathbf{v}}(t_r)/dt_r$

Note: $\displaystyle \frac{\partial \vec{\mathbf{v}}}{\partial t}\bigg|_{\vec{\mathbf{x}},t_r} = 0 \quad , \quad \frac{\partial s}{\partial t}\bigg|_{\vec{\mathbf{x}},t_r} = 0$

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Intermediate quantities

We need to calculate the two intermediate quantities:

$\nabla s|_{t,t_r}$ and $\partial s/\partial t_r|_{\vec{\mathbf{x}}}$.

- For the space derivative:

$$\nabla s\Big|_{t,t_r} = \frac{\partial s}{\partial x^\alpha}\Big|_{t,t_r} = \frac{\partial}{\partial x^\alpha}\Big|_{t_r}\left(r - \frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{c}\right) = \frac{\vec{\mathbf{r}}}{r} - \frac{\vec{\mathbf{v}}}{c} \tag{18}$$

- Starting from $s = r - \frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{c}$, we get the time derivative

$$\frac{\partial s}{\partial t_r}\Big|_{\vec{\mathbf{x}}} = \frac{\partial r}{\partial t_r}\Big|_{\vec{\mathbf{x}}} - \frac{\partial \vec{\mathbf{r}}}{\partial t_r}\Big|_{\vec{\mathbf{x}}}\cdot\frac{\vec{\mathbf{v}}}{c} - \frac{\vec{\mathbf{r}}}{c}\cdot\frac{\partial \vec{\mathbf{v}}}{\partial t_r}\Big|_{\vec{\mathbf{x}}} = -\frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{r} + \frac{v^2}{c} - \frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{a}}}{c} \tag{19}$$

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Result for the electric field $\vec{\mathbf{E}}$

- A straightforward (but messy) algebra gives

$$\frac{4\pi\epsilon_0}{q}\vec{\mathbf{E}}(\vec{\mathbf{x}},t) = \frac{1}{s^3}\left(\vec{\mathbf{r}} - \frac{r\vec{\mathbf{v}}}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \frac{1}{c^2 s^3}\left(\vec{\mathbf{r}} - \frac{r\vec{\mathbf{v}}}{c}\right)(\vec{\mathbf{r}}\cdot\vec{\mathbf{a}}) - \frac{r}{c^2 s^2}\vec{\mathbf{a}} \quad (20)$$

where the term in the top row does not depend on the acceleration, and the terms in the second row are linear in the acceleration $\vec{\mathbf{a}}$. There are no higher-order terms.

The acceleration-independent part

- The terms independent of acceleration are clearly the terms that will be present even when $\vec{\mathbf{a}} = 0$, and we already know the result for this. Indeed, we get

$$\vec{\mathbf{E}}(x,t)_{\vec{\mathbf{a}}=0} = \frac{q}{4\pi\epsilon_0}\frac{1}{\gamma^2 s^3}\vec{\mathbf{r}}_{\vec{\mathbf{v}}}\ , \tag{21}$$

  where $\vec{\mathbf{r}}_{\vec{\mathbf{v}}} \equiv \vec{\mathbf{r}} - \frac{r\vec{\mathbf{v}}}{c}$ corresponds to the position the charge *would have been* at time t, had it continued to travel with the same velocity as it had at the retarded time t_r.

- This is the same as the answer we obtained two lectures ago when we had no acceleration. Note that at $\vec{\mathbf{r}} \to \infty$, this component goes as $1/r^2$, and will not contribute to radiation.

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The acceleration-dependent part

- The acceleration-dependent part may be written as

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)_{\vec{\mathbf{a}}} = \frac{q}{4\pi\epsilon_0} \frac{1}{c^2 s^3} \vec{\mathbf{r}} \times (\vec{\mathbf{r}}_{\vec{\mathbf{v}}} \times \vec{\mathbf{a}}) \tag{22}$$

- At large r, this component goes as $1/r$, and hence will contribute to radiation. Thus, an accelerating charge radiates.

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Calculating magnetic field $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$

Problem

Starting with

$$\frac{4\pi\epsilon c^2}{q}\vec{\mathbf{B}}(\vec{\mathbf{x}}, t) = \nabla \times \left(\frac{\vec{\mathbf{v}}(t_r)}{s(t_r)}\right)\bigg|_t , \tag{23}$$

- Calculate $\vec{\mathbf{B}}$ and separate it into a component independent of $\vec{\mathbf{a}}$ and a component linear in $\vec{\mathbf{a}}$.
- Show that, the non-radiative part is

$$\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)_{\vec{\mathbf{a}}=0} = \frac{q}{4\pi\epsilon_0 c^2}\frac{1}{s^3\gamma^2}\,\vec{\mathbf{v}}(t_r) \times \vec{\mathbf{r}}(t_r) \tag{24}$$

- Show that, for the radiative component,

$$\vec{\mathbf{B}}_{\mathrm{rad}}(\vec{\mathbf{x}}, t) = \frac{\vec{\mathbf{r}}(t_r) \times \vec{\mathbf{E}}_{\mathrm{rad}}(\vec{\mathbf{x}}, t)}{r(t_r)c} \tag{25}$$

Poynting vector and radiation

Problem

From the expressions for $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$, calculate the Poynting vector. Hence argue that the accelerating charge indeed radiates nonzero energy to infinity.

Take-home message from this lecture

- For calculating $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$ from an accelerating charge, the quantities $\vec{\mathbf{r}}, r, s$ at different times and their derivatives need to be handled carefully.
- The acceleration gives rise to a component in $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ as well as $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$ that goes as $1/r$ at large r. Thus the charge radiates energy to infinity.
- The radiative parts of $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ as well as $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$ are linear in the acceleration $\vec{\mathbf{a}}(t_r)$, and vanish for a charge moving with uniform velocity, as expected.