Module III: Relativistic ED: applications

Lecture 22: Radiation from linear acceleration: Bremsstrahlung

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Outline

1. Linearly accelerated charge: radiation at small velocities
2. Bremsstrahlung radiation: large velocities

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Radiative electric and magnetic fields

- From the last lecture, we know the radiative components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields to be

$$\vec{\mathbf{E}}_{\rm rad}(\vec{\mathbf{x}}, t) = \frac{q}{4\pi\epsilon_0 c^2 s^3}\, \vec{\mathbf{r}} \times (\vec{\mathbf{r}}_{\vec{\mathbf{v}}} \times \vec{\mathbf{a}}) \quad (1)$$

$$\vec{\mathbf{B}}_{\rm rad}(\vec{\mathbf{x}}, t) = \frac{1}{rc}[\vec{\mathbf{r}} \times \vec{\mathbf{E}}(\vec{\mathbf{x}}, t)] \quad (2)$$

  where the quantities $\vec{\mathbf{r}}, r, s, \vec{\mathbf{v}}, \vec{\mathbf{a}}$ are all calculated at the retarded time t_r.

- Remember that t_r is to be determined from the solution to the implicit equation

$$t_r = t - r(t_r)/c\,. \quad (3)$$

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The low-velocity limit

- At low velocities ($|v| \ll c$), we can use $s \approx r$ and $\vec{\mathbf{r}}_{\vec{\mathbf{v}}} \approx \vec{\mathbf{r}}$ to get

$$\vec{\mathbf{E}}_{\text{rad}}(\vec{\mathbf{x}}, t) = \frac{q}{4\pi\epsilon_0 c^2 r^3}\, \vec{\mathbf{r}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{a}}) \quad (4)$$

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- Note that in this approximation, $t_r \approx t$. Therefore, all the quanities may be calculated at time t itself. The right hand side of the above equations can therefore be written in blue.

Problem

Show that the expressions for radiation fields are equivalent to an electric dipole. Find the relation between the electric dipole $\vec{\mathbf{p}}$ and the acceleration $\vec{\mathbf{a}}$ of the charge.

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Poynting vector at small velocities

- The Poynting vector is

$$\vec{\mathbf{N}}(\vec{\mathbf{x}}, t) = \frac{1}{\mu_0} \vec{\mathbf{E}}(\vec{\mathbf{x}}, t) \times \vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$$
$$= -\frac{1}{\mu_0} \left(\frac{q}{4\pi\epsilon_0} \right)^2 \frac{1}{c^5 r^5} (\vec{\mathbf{r}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{b}} \,, \qquad (6)$$

  where $\vec{\mathbf{b}} \equiv \vec{\mathbf{a}} \times \vec{\mathbf{r}}$.

- Since $\vec{\mathbf{b}} \perp \vec{\mathbf{r}}$, we get

$$(\vec{\mathbf{r}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{b}} = -|\vec{\mathbf{b}}|^2 \vec{\mathbf{r}} = -|\vec{\mathbf{a}} \times \vec{\mathbf{r}}|^2 \vec{\mathbf{r}} \,. \qquad (7)$$

  When θ_{ar} is the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{r}}$, this quantity becomes $-|\vec{\mathbf{a}}|^2 r^2 \sin^2\theta_{ar}\, \vec{\mathbf{r}}$.

- The Poynting vector is then

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Power radiated at small velocities

- We take the direction of acceleration to be the z-axis. Then $\theta_{ar} = \theta$, the polar angle of the spherical polar coordinates.
- The power radiated in a solid angle $d\Omega$ is then

$$\frac{dU}{dt} = (\vec{\mathbf{N}} \cdot \hat{\mathbf{r}}) r^2 d\Omega = \frac{q^2 |\vec{\mathbf{a}}|^2}{16\pi^2 \epsilon_0 c^3} \sin^2\theta \, d\Omega \tag{9}$$

  Note that the angular dependence is $\sin^2\theta$, where θ is the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{r}}$. The radiation is thus peaked in the direction normal to the acceleration.
- Integrating over all angles, the total power radiated is

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Rate of energy loss of the charge

- This power radiated can also be looked upon as the rate of energy loss of the particle. Strictly speaking, the latter quantity is (dU/dt_r). However since $s \approx r$, here we have $\partial/\partial t = \partial/\partial t_r$.
- The rate of energy loss (at small velocities) is then

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1. Linearly accelerated charge: radiation at small velocities

2. Bremsstrahlung radiation: large velocities

Accelerating charge in a linear motion

- For a linear motion, $\vec{\mathbf{v}} \parallel \vec{\mathbf{a}}$, and therefore $\vec{\mathbf{r}}_v \times \vec{\mathbf{a}} = \vec{\mathbf{r}} \times \vec{\mathbf{a}}$. As a result,

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0 c^2 s^3}[\vec{\mathbf{r}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{a}})] \ , \quad \vec{\mathbf{B}} = \frac{q}{4\pi\epsilon_0 c^3 s^3} r\,(\vec{\mathbf{a}} \times \vec{\mathbf{r}}) \qquad (12)$$

- A straightforward calculation gives the time-averaged Poynting vector

$$\vec{\mathbf{N}}(\vec{\mathbf{x}}, t) = \frac{q^2}{16\pi^2\epsilon_0 c^3 s^6}|\vec{\mathbf{a}}|^2 r^4 \sin^2\theta\, \hat{\mathbf{r}} \qquad (13)$$

- Using $s = r - \frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{c} = r(1 - \frac{v}{c}\cos\theta)$, we get the angular dependence of the power radiated as

$$\frac{dU}{dt} = \frac{q^2}{16\pi^2\epsilon_0 c^3 (1 - \frac{v}{c}\cos\theta)^6}|\vec{\mathbf{a}}|^2 \sin^2\theta\, d\Omega \qquad (14)$$

The peak of the radiation thus moves more and more in the forward direction (but not at $\theta = 0$) as the velocity increases, since the factor $(1 - \frac{v}{c}\cos\theta)^6$ in the denominator decreases sharply. See Mathematica file

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- This needs to be contrasted with the rate of loss of energy of the charge, which is

$$\frac{dU}{dt_r} = \frac{s}{r}\frac{dU}{dt} = \frac{q^2|\vec{\mathbf{a}}|^2}{16\pi^2\epsilon_0 c^3} \int \frac{\sin^2\theta}{(1-\frac{v}{c}\cos\theta)^5} 2\pi d\cos\theta \qquad (16)$$

  Note that the power of the factor $(1-\frac{v}{c}\cos\theta)$ is different in the two expressions.

- This radiation would therefore make the charge lose energy and decrease its speed. Hence it is called "breaking radiation" or *bremsstrahlung*.

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Radiated energy for constant acceleration

Problem

Accelerating charge:

(i) By an explicit calculation, show that the magnetic field $\vec{\mathbf{B}}$ from an accerating charge reduces to the form $\vec{\mathbf{B}} = \vec{\mathbf{r}} \times \vec{\mathbf{E}}/(rc)$ where $\vec{\mathbf{r}} = \vec{\mathbf{x}}(t) - \vec{\mathbf{x}}'(t_r)$.

(ii) Calculate the total energy radiated in Bremsstrahlung when the speed of a charge q decreases at a constant rate a, from v_0 to 0.

Particle losing energy at a constant rate

Problem

A relativistic particle is losing energy at a constant rate $\mathcal{R} = dE/dt'$ while moving through a material in a straight line. In the process, the speed of the particle decreases from $v = 0.9c$ to $v = 0$.

(i) Plot the power radiated as a function of $\cos\theta$ (the angle between $\vec{\mathbf{v}}$ and $\vec{\mathbf{r}}$), when the speed of the particle is $v = 0.9c$, $v = 0.5c$ and $v = 0.1c$ (on the same plot, showing the relative magnitudes, in appropriate units).

(ii) Calculate the total energy radiated by the particle in the form of Bremsstrahlung radiation. You may need to integrate numerically.

Take-home message from this lecture

- The radiation from an accelerating charge is peaked in the direction transverse to its acceleration. The peak becomes sharper with increasing velocity of the particle.
- The power radiated by the accelerating charge and the rate of loss of energy of the accelareting charge are different quantities, the distinction being more significant at higher velocities of the charge.
- An accelerating particle undergoing a linear motion does not radiate exactly along its direction of motion. However at large speeds, the energy radiated in the forward direction predominates.