Module III: Relativistic ED: applications

Lecture 23: Radiation from circular orbits: Synchrotron

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Outline

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$\vec{\mathbf{E}}$, $\vec{\mathbf{B}}$ and $\vec{\mathbf{N}}$ from an accelerated charge

- The radiative components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields can be written as

$$\vec{\mathbf{E}}_{\rm rad}(\vec{\mathbf{x}}, t) = \frac{q}{4\pi\epsilon_0 c^2 s^3}\,\vec{\mathbf{b}}_{\vec{\mathbf{v}}} \times \vec{\mathbf{r}} \tag{1}$$

$$\vec{\mathbf{B}}_{\rm rad}(\vec{\mathbf{x}}, t) = \frac{1}{rc}[\vec{\mathbf{r}} \times \vec{\mathbf{E}}(\vec{\mathbf{x}}, t)] = \frac{q}{4\pi\epsilon_0 c^3 s^3 r}\vec{\mathbf{r}} \times (\vec{\mathbf{b}}_{\vec{\mathbf{v}}} \times \vec{\mathbf{r}}) \tag{2}$$

  where $\vec{\mathbf{b}}_{\vec{\mathbf{v}}} \equiv \vec{\mathbf{a}} \times \vec{\mathbf{r}}_v$. Note that the quantities $\vec{\mathbf{r}}, r, s, \vec{\mathbf{v}}, \vec{\mathbf{a}}$ are all calculated at the retarded time t_r.

- The Poynting vector is

$$\begin{aligned}\vec{\mathbf{N}}(\vec{\mathbf{x}}, t) &= \frac{q^2}{16\pi^2\epsilon_0 c^3 s^6 r}(\vec{\mathbf{b}}_{\vec{\mathbf{v}}} \times \vec{\mathbf{r}}) \times [\vec{\mathbf{r}} \times (\vec{\mathbf{b}}_{\vec{\mathbf{v}}} \times \vec{\mathbf{r}})] \\ &= \frac{q}{16\pi^2\epsilon_0 c^3 s^6}|\vec{\mathbf{b}}_v \times \vec{\mathbf{r}}|^2\, \hat{\mathbf{r}}\,.\end{aligned} \tag{3}$$

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Power radiated from an accelerated charge

- The power radiated in solid angle $d\Omega$ is

$$\frac{dU}{dt} = \frac{q^2}{16\pi^2\epsilon_0 c^3 s^6} r^2 |\vec{\mathbf{b}}_v \times \vec{\mathbf{r}}|^2 d\Omega \tag{4}$$

- We therefore need to calculate the quantity $|\vec{\mathbf{b}}_v \times \vec{\mathbf{r}}|^2$. For this, we should first define our notation carefully.

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Coordinates of the source

- Let the charge be moving in a circle of radius A in the x-y plane with an angular frequency ω_0.
- Let the coordinates of the rotating charge be

$$\vec{\mathbf{x}}'(t_r) = A\,(\cos\varphi, \sin\varphi, 0)\ , \tag{5}$$

  where $\varphi(t_r) = \omega_0 t_r + \varphi_0$.
- The velocity and acceleration in terms of the coordinate φ is:

$$\begin{aligned}\vec{\mathbf{v}}(t_r) &= A\omega_0(-\sin\varphi, \cos\varphi, 0) \\ \vec{\mathbf{a}}(t_r) &= A\omega_0^2(-\cos\varphi, -\sin\varphi, 0)\ .\end{aligned} \tag{6}$$

- We are interested in the power radiated at a distance $r \gg A$ from the charge. We shall therefore neglect all terms suppressed by A/r. In particular, we take $\vec{\mathbf{r}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}'(t_r) \approx \vec{\mathbf{x}}$.

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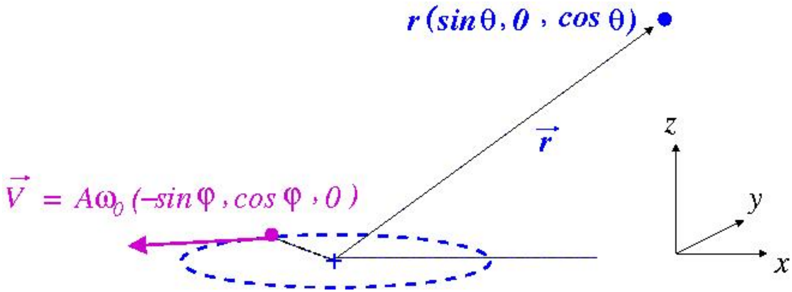
Coordinates of the observation point



- The vector $\vec{\mathbf{r}}$ can in general be any vector of radius r. I.e.

$$\vec{\mathbf{r}} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{7}$$

- However since the problem has azimuthal symmetry, we choose our x-axis such that the observation point is in the x-z plane. That is, $\phi = 0$, and

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- This choice of x-axis determines the value of φ_0.

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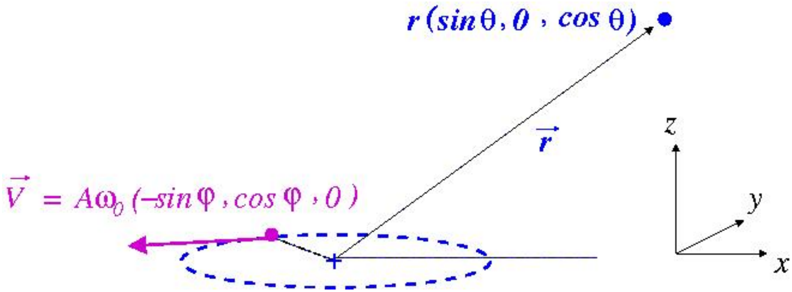

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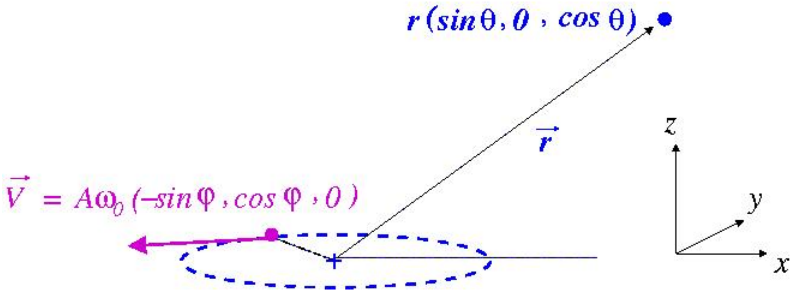

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Intermediate steps towards calculating radiated power

- We then calculate

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\vec{\mathbf{r}}_{\vec{\mathbf{v}}} &= r\left(\sin\theta + \frac{A\omega_0}{c}\sin\varphi, -\frac{A\omega_0}{c}\cos\varphi, \cos\theta\right) \\
\vec{\mathbf{b}}_{\vec{\mathbf{v}}} &= A\omega_0^2 r\left(-\cos\theta\sin\varphi, \cos\theta\cos\varphi, \frac{A\omega_0}{c} + \sin\theta\sin\varphi\right) \\
\vec{\mathbf{b}}_{\vec{\mathbf{v}}} \times \vec{\mathbf{r}} &= A\omega_0^2 r^2\left(\cos^2\theta\cos\varphi, \frac{A\omega_0}{c}\sin\theta + \sin\varphi, -\sin\theta\cos\theta\cos\varphi\right)
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- And hence,

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Net radiated power

- The radiated power in a solid angle $d\Omega$ is

$$\frac{dU}{dt} = \frac{q^2 A^2 \omega_0^4}{16\pi^2 \epsilon_0 c^3 \left(1 + \frac{A\omega_0}{c} \sin\theta \sin\varphi\right)^6} \times \left[\left(\frac{A\omega_0}{c} \sin\theta + \sin\varphi\right)^2 + \cos^2\theta \cos^2\varphi\right] d\Omega \quad (11)$$

where we have used $s = r - \frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{c} = r(1 + \frac{A\omega_0}{c} \sin\theta \sin\varphi)$.

- In order to calculate the time-averaged radiated power, we have to integrate over one cycle of φ and normalize :

$$\left\langle \frac{dU}{dt} \right\rangle = \int \frac{d\varphi}{2\pi} \frac{dU}{dt} \quad (12)$$

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Power radiated in the x-y plane

- In the x-y plane, i.e. at $\sin\theta = 1$, we have

$$\frac{dU}{dt} = \frac{q^2 A^2 \omega_0^4}{16\pi^2 \epsilon_0 c^3} \frac{\left(\frac{v}{c} + \sin\varphi\right)^2}{\left(1 + \frac{v}{c}\sin\varphi\right)^6} d\Omega \tag{13}$$

  where we have used $v = A\omega_0$.

- When $v \approx c$, the radiated power in the x-y plane is enhanced by the factor of $(1 + \frac{v}{c}\sin\varphi)^6$ in the denominator, due to the times when the charge is at $\varphi(t_r) = -\pi/2$, i.e. $\sin\varphi = -1$.
- However at a given observation point, this high radiation burst lasts only during $\sin\varphi$ variation $(-v/c) \to (-1) \to (-v/c)$. See Mathematica file
- For other positions of the charge, the power radiated is

$$\frac{dU}{dt} \sim \frac{q^2 A^2 \omega_0^4}{16\pi^2 \epsilon_0 c^3} d\Omega \tag{14}$$

- Thus in the x-y plane, the synchrotron radiation occurs in small bursts, one burst per cycle of the charge.

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Power radiated along the z-axis

- Along the z-axis, i.e. at $\sin\theta = 0$, we have

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- Thus, there is no time dependence for the power radiated along z-axis, and also no enhancement.
- The nature of synchrotron radiation – both intensity and time structure – thus varies drastically depending on the angle made by $\vec{r}$ with the plane of the motion of the charge.
- The large enhancement in the x-y plane implies that most of the power is emitted in this plane.

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Dependence of radiated power on φ and θ

Problem

(i) Plot the φ-dependence of dU/dt in the x-y plane, for $v = 0.5c$, $v = 0.9c$ and $v = 0.99c$ on the same plot in appropriate units. (You may have to use a logarithmic scale.) Comment on the this angular dependence.

(ii) Plot the average power radiated by the charge as a function of θ for $v = 0.5c$, $v = 0.9c$ and $v = 0.99c$ on the same plot in appropriate units. (You may have to use a logarithmic scale.) Comment on the this angular dependence.

Outline

Instantaneous power at $\varphi(t_r) = 0$

- To study the instantaneous pattern of radiated power, we rotate our frame in the x-y plane such that the positive x-axis is along the particle, i.e. $\varphi(t_r) = 0$.
- This rotation takes $\varphi(t_r) \to 0$, and also takes the azimuthal angle of the observation point, ϕ, to the original value of $-\varphi(t_r)$.
- The radiated power is then obtained by substituting $\varphi \to -\phi$ in our earlier expression for radiated power. Note that since "instantaneous" here refers to the time t_r, we are interested in the quantity dU/dt_r, i.e. power radiated as a function of the coordinates of the source.

$$\frac{dU}{dt_r} = \frac{s}{r}\frac{dU}{dt} = \frac{q^2 A^2 \omega_0^4}{16\pi^2 \epsilon_0 c^3 \left(1 - \frac{v}{c}\sin\theta\sin\phi\right)^5} \times \left[\left(\frac{v}{c}\sin\theta - \sin\phi\right)^2 + \cos^2\theta\cos^2\phi\right] d\Omega \quad (16)$$

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$$\frac{dU}{dt_r} = \frac{s}{r}\frac{dU}{dt} = \frac{q^2 A^2 \omega_0^4}{16\pi^2 \epsilon_0 c^3 \left(1 - \frac{v}{c}\sin\theta\sin\phi\right)^5} \times \left[\left(\frac{v}{c}\sin\theta - \sin\phi\right)^2 + \cos^2\theta\cos^2\phi\right] d\Omega \quad (16)$$

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Power peaking in the forward direction

- The power radiated may be written in a convenient form

$$\frac{dU}{dt_r} = \frac{q^2 A^2 \omega_0^4}{16\pi^2\epsilon_0 c^3 \left(1 - \frac{v}{c}\sin\theta\sin\phi\right)^3}\left[1 - \frac{\sin^2\theta\ \cos^2\phi}{\gamma^2\left(1 - \frac{v}{c}\sin\theta\sin\phi\right)^2}\right] d\Omega \tag{17}$$

- The large powers of $[1 - (v/c)\sin\theta\sin\phi]$ in the denominator cause the radiation to peak strongly when $v \approx c$, $\sin\theta \approx 1$, and $\sin\phi \approx 1$.
- Thus the radiation is peaked in the forward direction ($\sin\theta \approx 1$, and $\sin\phi \approx 1$) in the relativistic case ($v \approx c$).
- The angular dependence could be made more explicit by choosing the "forward" direction as the new $\tilde{z}$ axis. This may be done by using the rotation which makes

$$\tilde{z} = y\,, \quad \tilde{y} = x\,, \quad \tilde{x} = z\,.$$

In this new coordinate system,

$$\vec{\mathbf{v}} = v(0,0,1)\,, \quad \vec{\mathbf{r}} = r(\sin\tilde{\theta}\cos\tilde{\phi}, \sin\tilde{\theta}\sin\tilde{\phi}, \cos\tilde{\theta})\,. \tag{18}$$

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The coordinate transformations

- We shall now write down our expression for radiated power in this new coordinate system.
- Since scalar products are the same in both frames,

$$\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = vr \sin\theta \sin\phi = vr \cos\tilde{\theta} \,, \tag{19}$$

$$\begin{aligned} |\vec{\mathbf{r}} \times \vec{\mathbf{v}}|^2 &= r^2 v^2 (\cos^2\theta + \sin^2\theta \cos^2\phi) \\ &= r^2 v^2 (\sin^2\tilde{\theta} \sin^2\tilde{\phi} + \sin^2\tilde{\theta} \cos^2\tilde{\phi}) \,. \end{aligned} \tag{20}$$

These relations yield

$$\sin\theta \sin\phi = \cos\tilde{\theta} \,, \quad \sin^2\theta \cos^2\phi = \sin^2\tilde{\theta} \cos^2\tilde{\phi} \tag{21}$$

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Quantifying the forward-peaked nature

- Clearly, when $v \approx c$ and $\cos\tilde{\theta} \approx 1$ (i.e. the forward direction), the radiated power is enhanced.
- Since the enhancement is only for small $\tilde{\theta}$, we may expand $\cos\tilde{\theta} \approx 1 - \tilde{\theta}^2/2$ and $\sin\tilde{\theta} \approx \tilde{\theta}$ to get

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- In the limit $v = c$, this becomes

$$\frac{dU}{dt_r} = \frac{q^2 A^2 \omega_0^4}{16\pi^2 \epsilon_0 c^3} \frac{8}{\tilde{\theta}^6} \left[1 - \frac{4\cos^2\tilde{\phi}}{\gamma^2 \tilde{\theta}^2}\right] d\tilde{\Omega} \tag{24}$$

- Thus the enhancement at small $\tilde{\theta}$ is extremely high in this limit, but also falls off sharply as $\tilde{\theta}$ increases.
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- In the limit $v \approx c$, i.e. large boost γ, using $v/c = 1 - 1/(2\gamma^2)$, and neglecting the doubly suppressed quantity $\tilde{\theta}^2/\gamma^2$, one gets

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- Clearly, larger the boost, sharper the forward peak.

Problem

Plot dU/dt_r as a function of $\tilde{\theta}$, for two values of $\tilde{\phi} : 0, \pi/2$ and three values of $\gamma : 1, 10, 100$. Comment on your results.

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Dependence of radiated power on the boost

- In the limit $v \approx c$, i.e. large boost γ, using $v/c = 1 - 1/(2\gamma^2)$, and neglecting the doubly suppressed quantity $\tilde{\theta}^2/\gamma^2$, one gets

$$\begin{aligned}\frac{dU}{dt_r} &= \frac{q^2 A^2 \omega_0^4}{16\pi^2\epsilon_0 c^3 \left(\frac{1}{2\gamma^2} + \frac{\tilde{\theta}^2}{2}\right)^3} \left[1 - \frac{\tilde{\theta}^2 \, \cos^2\tilde{\phi}}{\gamma^2 \left(\frac{1}{2\gamma^2} + \frac{\tilde{\theta}^2}{2}\right)^2}\right] d\tilde{\Omega} \\ &= \frac{q^2 A^2 \omega_0^4 \gamma^6}{2\pi^2\epsilon_0 c^3 \left(1 + \tilde{\theta}^2\gamma^2\right)^3} \left[1 - \frac{4\gamma^2\tilde{\theta}^2 \, \cos^2\tilde{\phi}}{\left(1 + \tilde{\theta}^2\gamma^2\right)^2}\right] d\tilde{\Omega} \,. \qquad (25)\end{aligned}$$

- Clearly, larger the boost, sharper the forward peak.

Problem

Plot dU/dt_r as a function of $\tilde{\theta}$, for two values of $\tilde{\phi} : 0, \pi/2$ and three values of $\gamma : 1, 10, 100$. Comment on your results.

Total radiated power

- In order to find the total rate of radiated power, we integrate over the solid angle $d\tilde{\Omega} = d\cos\tilde{\theta}d\tilde{\phi}$.
- Since any significant contribution to the radiated power will only be from small $\tilde{\theta}$ (i.e. with $\tilde{\theta} \lesssim 1/\gamma$), in order to get the integral in a closed form, we take the range of $\tilde{\theta}$ to be from 0 to ∞.
- The integration gives the total power radiated to be

$$\begin{aligned} \frac{dU}{dt_r} &= \frac{q^2 A^2 \omega_0^4 \gamma^6}{2\pi^2 \epsilon_0 c^3} \left[\frac{2\pi}{4\gamma^2} - \frac{4\pi}{24\gamma^2} \right] \\ &= \frac{q^2 A^2 \omega_0^4 \gamma^4}{6\pi \epsilon_0 c^3} \end{aligned} \qquad (26)$$

- The power loss is thus proportional to the fourth power of γ, or if one wants to compare two particles with the same energy, the power loss goes as $1/m^4$. Therefore, it is difficult to increase the energy of an electron accelerator beyond a certain energy. The accelerators with protons can go to much higher energies. (E.g., the energy of electrons in LEP was $\sim$ 100 GeV, while the protons in the LHC, in the same ring, can be accelerated to a few TeV.)

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Calculating radiated power

Problem

(i) Show the angular distribution of the radiated power as a SphericalPlot3D (Mathematica notation) for $\gamma = 1, 10, 100$ in appropriate units. Comment on the angular dependence.

(ii) Numerically perform the angular integration of dU/dt_r (without the approximation of small $\tilde{\theta}$) to calculate the total power radiated for $\gamma = 1, 10, 100$ in appropriate units. Compare the answers with those obtained analytically by using $\cos\tilde{\theta} \approx 1 - \tilde{\theta}^2$.

Outline

Synchrotron radiation burst for $v \approx c$

- We have seen that in the x-y plane, the synchrotron radiation appears in spurts with frequency ω_0.
- The typical duration of the burst is the time it takes for φ to go $(-v/c) \to (-1) \to (-v/c)$:

$$\Delta t' \sim 2\frac{1-\frac{v}{c}}{\omega_0} \sim \frac{1}{\omega_0 \gamma^2} \tag{27}$$

where we have used

$$1/\gamma^2 = (1+v/c)(1-v/c) \approx 2(1-v/c)\ . \tag{28}$$

- Since the burst occurs for a short duration $\Delta t' \sim (\omega_0\gamma^2)^{-1}$, by uncertainty theorem it consists of a frequency spread as high as $\Delta\omega' \sim \omega_0\gamma^2$.

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Frequency of observed synchrotron pulse

- For an observation point in the *x*-*y* plane, since the source is travelling towards it, the frequencies undergo a Doppler blue-shift by a factor γ.
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X-ray production

Problem

Electrons are to be accelerated in a circle of 10 cm radius in order to generate X-rays of energy $\sim 1keV$.

(i) Estimate the speed the electrons need to be accelerated to, and hence the frequency ω_0.

(ii) What is the kinetic energy of the electrons ? hence, how much magnetic field will be needed to keep the electrons in this orbit ?

Take-home message from this lecture

- The radiation from a charge in a circular orbit is azimuthally symmetric on average, but has a strong dependence on the angle made by $\vec{\mathbf{r}}$ with the axis of the motion
- The radiated power is enhanced in the x-y plane for small bursts. These bursts may be used to make high-frequency (X-ray) radiation sources, the frequencies scaling as γ^3.
- The rate of loss of energy of a charge due to synchrotron scales as m^{-4}, making it more difficult to build accelerators with lighter particles like electrons.