

Module III: Relativistic ED: applications

Lecture 24: Radiation reaction force

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- 1 Force of an accelerating charge on itself: small \vec{v}
- 2 Radiation damping in ultrarelativistic case

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Loss of energy from the action of a force

- We know that an accelerating charge radiates, and hence loses energy.
- For an object to lose energy, it must have been acted upon by a force.
- Here, this is a force exerted by the charge on itself, through the creation of an EM field that acts back on the charge
- This is the “radiation reaction force”.

Work done on the charge by the radiation

- We have seen that, for small \vec{v} , the rate of energy loss of a particle is

$$\frac{dU}{dt_r} = \frac{q^2}{6\pi\epsilon_0 c^3} |\vec{a}|^2 \quad (1)$$

- Integrating over time, this should give the negative of the work done on the charge. I.e. if \vec{f}_{rad} is the force on the charge due to the radiation,

$$\int_{t_1}^{t_2} \vec{f}_{\text{rad}} \cdot \vec{v} dt = -\frac{q^2}{6\pi\epsilon_0 c^3} \int_{t_1}^{t_2} \vec{a} \cdot \vec{a} dt \quad (2)$$

(Since we are dealing with small \vec{v} , there is no difference between t_r and t .)

- Integrating by parts gives

$$\int_{t_1}^{t_2} \vec{f}_{\text{rad}} \cdot \vec{v} dt = -\frac{q^2}{6\pi\epsilon_0 c^3} \vec{a} \cdot \vec{v} \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{q^2}{6\pi\epsilon_0 c^3} \frac{d\vec{a}}{dt} \cdot \vec{v} dt \quad (3)$$

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An average force

- The first term on the RHS vanishes for periodic motion, if there are an integer number of periods between t_1 and t_2 .
- If the charge is accelerating for a finite time, a proper choice of t_1 and t_2 would make this term vanish.
- For uncorrelated \vec{v} and \vec{a} , this term would vanish for an average over random (t_1, t_2) pairs.
- In this sense, on an average, we may equate

$$\vec{f}_{\text{rad}} = \frac{q^2}{6\pi\epsilon_0 c^3} \frac{d\vec{a}}{dt} \quad (4)$$

- The radiation reaction force is thus proportional to the rate of change of acceleration, $d\vec{a}/dt$, also called as “jerk”.
- Note that this expression is only valid as an average over time. This is because in addition to emitting energy at infinity, the charge is also exchanging energy with the \vec{E} and \vec{B} fields, which averages out in the long run.

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Another interpretation

- The radiation reaction force may be looked upon as the net force exerted by different parts of the charge on each other, however it does not need the charge to be of a finite size.
- For example, one may consider a point charge (an electron, for example) to be a spherical charge distribution of radius r_0 , and calculate the $\vec{\mathbf{E}}$ field at the position of a charge element due to all the others. This will give the force on that element. Adding up all the charge elements, in the limit $r_0 \rightarrow 0$, we'll get the force on the charge. (See Panofsky-Phillips)

The “jerk” force

- This is one of the rare instances where the third derivative of position enters a force equation.
- Though we used the word “rare” here, it is a feature of many practical situations, like the behaviour of electrons inside a material, as we’ll see in the next lecture. So this is indeed an important phenomenon.
- Indeed, **this force would make an electron orbiting a nucleus lose energy and fall inside the nucleus**. This quandary led Bohr to postulate the quantization of angular momentum in his atomic model, one of the first foundations of Quantum Mechanics.

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Validity of the \vec{f}_{rad} expression

- If radiation reaction force were the only force acting on a particle, the equation of motion would become

$$m\vec{a} = \frac{q^2}{6\pi\epsilon_0 c^3} \frac{d\vec{a}}{dt} \quad (5)$$

- This gives an **unstable solution**, since as soon as \vec{a} is non-vanishing, it tends to blow up.
- Therefore, the approximation of \vec{f}_{rad} as an average force is **valid only when there is an external force that determines the acceleration of the particle**.
- Another way of seeing this is that if the acceleration is dominated by \vec{f}_{rad} itself, **\vec{v} and \vec{a} will not be uncorrelated**, so the “averaging out” of the first term on the RHS of eq. (3) is not valid.

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Validity conditions with external EM fields

- Let the external force be the Lorentz force $\vec{F}_{\text{ext}} = q(\vec{E} + \vec{v} \times \vec{B})$.
- Then the equation of motion is

$$m\vec{a} = q\vec{E} + q\vec{v} \times \vec{B} + \frac{q^2}{6\pi\epsilon_0 c^3} \frac{d\vec{a}}{dt}. \quad (6)$$

- We want the last term, $\vec{f}_{\text{rad}} \equiv q^2(d\vec{a}/dt)/(6\pi\epsilon_0 c^3)$, to be small compared to the first two. To determine \vec{f}_{rad} , we first differentiate the above equation in the limit $\vec{f}_{\text{rad}} = 0$, to get

$$m\dot{\vec{a}} \approx q(d\vec{E}/dt) + q\vec{a} \times \vec{B} + q\vec{v} \times (d\vec{B}/dt). \quad (7)$$

- In the above expression, we can substitute

$$d\vec{E}/dt \sim i\omega\vec{E}, \quad \vec{a} \sim \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) = \frac{q}{m}\vec{E} \quad (8)$$

where in the last step we have used the approximation $\vec{v} \approx 0$, which makes the last term on RHS vanish. This gives

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Interpretation of the validity conditions

- The radiation reaction force is thus

$$\vec{f}_{\text{rad}} \sim \frac{q^3 \omega}{6\pi \epsilon_0 m c^3} \vec{E} + \frac{q^4}{6\pi \epsilon_0 m^2 c^3} \vec{E} \times \vec{B} \quad (10)$$

- Since we require $\vec{f}_{\text{rad}} \ll q\vec{E} + \vec{v} \times \vec{B} \approx q\vec{E}$ in our chosen frame, the two terms above should be much smaller than $q\vec{E}$.
- The first condition requires

$$\frac{q^3 \omega}{6\pi \epsilon_0 m c^3} \ll q, \quad \text{i.e.} \quad \lambda = \frac{c}{\omega} \gg \frac{q^2}{6\pi \epsilon_0 m c^2}. \quad (11)$$

This implies that the wavelength of the EM radiation should be much greater than the classical radius of the charge.

- The second condition requires

$$\frac{q^4}{6\pi \epsilon_0 m^2 c^3} |\vec{B}| \ll q, \quad \text{i.e.} \quad |\vec{B}| \ll \frac{m^2 c^3}{q^3 6\pi \epsilon_0}, \quad (12)$$

i.e. the magnetic field should not be much larger than some critical value. This essentially corresponds to saying that the external EM fields should not be too large.

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- Since we require $\vec{f}_{\text{rad}} \ll q\vec{E} + \vec{v} \times \vec{B} \approx q\vec{E}$ in our chosen frame, the two terms above should be much smaller than $q\vec{E}$.
- The first condition requires

$$\frac{q^3 \omega}{6\pi \epsilon_0 m c^3} \ll q, \quad \text{i.e.} \quad \lambda = \frac{c}{\omega} \gg \frac{q^2}{6\pi \epsilon_0 m c^2}. \quad (11)$$

This implies that the wavelength of the EM radiation should be much greater than the classical radius of the charge.

- The second condition requires

$$\frac{q^4}{6\pi \epsilon_0 m^2 c^3} |\vec{B}| \ll q, \quad \text{i.e.} \quad |\vec{B}| \ll \frac{m^2 c^3}{q^3 6\pi \epsilon_0}, \quad (12)$$

i.e. the magnetic field should not be much larger than some critical value. This essentially corresponds to saying that the external EM fields should not be too large.

- In the next class, we'll see how a term of the form of $\vec{\mathbf{f}}_{\text{rad}}$ can help us understand the propagation of EM radiation through matter

- 1 Force of an accelerating charge on itself: small \vec{v}
- 2 Radiation damping in ultrarelativistic case

The 4-vector for force

- We have already seen that the acceleration 4-vector is

$$a^k \equiv c \frac{du^k}{ds} = c \frac{d}{ds}(\gamma c, \gamma \vec{v}) \quad (13)$$

- The force 4-vector can be simply defined as

$$f^k \equiv ma^k = c \frac{dp^k}{ds} \quad (14)$$

- Recall $a^k u_k = 0$. Therefore, we have

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Covariant analogue of the EM force equation

- The covariant analogue of

$$m\vec{a} = q(\vec{E} + \vec{v} \times \vec{B}) + \vec{f}_{\text{rad}} \quad (16)$$

is

$$mc \frac{du^k}{ds} = qF^{km}u_m + f_{\text{rad}}^k \quad (17)$$

- We need f^k that corresponds to

$$\vec{f}_{\text{rad}} = \frac{q^2}{6\pi\epsilon_0 c^3} \frac{d\vec{a}}{dt} \quad \text{when } \vec{v} \rightarrow 0 \quad (18)$$

and satisfies

$$f_{\text{rad}}^k u_k = 0. \quad (19)$$

- Clearly, the naive guess

$$f_{\text{naive}}^k = \frac{q^2}{6\pi\epsilon_0 c^3} c^2 \frac{d^2 u^k}{ds^2} \quad (20)$$

does not work since here $f_{\text{naive}}^k u_k \neq 0$.

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The covariant definition of the radiation force

- In order to obtain the covariant \mathbf{f}_{rad} , we can use the ansatz

$$\mathbf{f}_{\text{rad}}^k = \mathbf{f}_{\text{naive}}^k + \alpha \mathbf{u}^k, \quad (21)$$

which clearly will continue to satisfy the $\vec{\mathbf{v}} \rightarrow 0$ limit, and the appropriate value of α will help us satisfy the second condition too.

- Requiring $\mathbf{f}_{\text{rad}}^k \mathbf{u}_k = 0$, we get

$$\mathbf{f}_{\text{rad}}^k = \frac{q^2}{6\pi\epsilon_0 c} \left[\frac{d^2 \mathbf{u}^k}{ds^2} - \mathbf{u}^k \mathbf{u}^m \frac{d^2 \mathbf{u}_m}{ds^2} \right]. \quad (22)$$

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f_{rad}^k for a particle in an EM field

- The covariant Lorentz force equation is

$$\frac{du^k}{ds} = \frac{q}{mc} F^{km} u_m. \quad (23)$$

- Differentiating gives

$$\begin{aligned} \frac{d^2 u^k}{ds^2} &= \frac{q}{mc} \left[\frac{\partial F^{km}}{\partial s} u_m + F^{km} \frac{du_m}{ds} \right] \\ &= \frac{q}{mc} \left[\frac{\partial F^{km}}{\partial x^n} u^n u_m + \frac{q}{mc} F_{mn} u^n F^{km} \right] \end{aligned} \quad (24)$$

- Using the definition of f_{rad} obtained earlier, we get

$$f_{\text{rad}}^k = \frac{q^3}{6\pi\epsilon_0 mc^2} \left[\frac{\partial F^{km}}{\partial x^n} u^n u_m + \frac{q}{mc} F_{mn} F^{km} u^n + \frac{q}{mc} (F_{\ell n} u^n) (F^{m\ell} u_m) u^k \right]. \quad (25)$$

(Note: one term vanishes due to antisymmetry of F.)

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Leading contribution in the ultra-relativistic limit

- Each factor of u in a term comes with a factor of γ . Therefore at large velocities, the last term, the one with three u factors, dominates. The **leading term at large velocities** is thus

$$\begin{aligned} f_{\text{rad}}^k &= \frac{q^4}{6\pi\epsilon_0 m^2 c^3} (F_{\ell n} u^n) (F^{m\ell} u_m) u^k \\ &= \frac{q^2}{6\pi\epsilon_0 c^3} a_\ell a^\ell u^k \end{aligned} \quad (26)$$

- Note that the reaction force in this case is **proportional to the square of the acceleration**.

Rate of loss of energy of an ultra-relativistic charge

Problem

The radiation reaction force on an ultra-relativistic charge is $\vec{f}_{\text{rad}} \propto \gamma^2 \propto \mathcal{E}^2$, where \mathcal{E} is the kinematic energy of the charge.

Starting from $d\mathcal{E}/dx = -k(x)\mathcal{E}^2$, determine how the kinematic energy of the charge will change as a function of time / distance travelled.

Problem

Find the conditions of validity of the radiation reaction force in the ultra-relativistic limit.

Problem

Can one obtain the small- \vec{v} limit of the radiation reaction force starting from the relativistic expression ?

Take-home message from this lecture

- The EM fields radiated by an accelerating charge result in a reaction force on the charge, thus slowing it down.
- In the small-velocity limit, the reaction force is proportional to the rate of change of acceleration. However this approximation is valid only when the particle is in a non-vanishing EM field which is not too large.
- In the ultra-relativistic case, the reaction force is proportional to the square of the acceleration.