Module III: Relativistic ED: applications

Lecture 25: EM radiation passing through matter

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Outline

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1. Interactions of EM fields with electrons
2. Scattering of EM wave by a free electron
3. Scattering of an EM wave by a bound electron
4. Absorption by a bound electron
5. Refractive index: collective scattering by electrons

Electron in a harmonic potential

- When an EM wave passes through matter, its propagation may be understood at the microscopic level by looking at its interactions with the electrons in the material.
- A simple model that helps in understanding many phenomena is that of electrons in a harmonic potential that are acted upon by the EM field and the radiation reaction force.
- This model will help us understand Thomson scattering, resonant scattering of light, Rayleigh scattering, absorption and refractive index of materials.
- The harmonic force $\vec{\mathbf{F}} = -k\vec{\mathbf{x}} = -m\omega_0^2\vec{\mathbf{x}}$, provided by the binding of the electrons with the atoms, is the dominant force and gives rise to the leading order equation of motion

$$m\vec{\mathbf{a}} = -m\omega_0^2\vec{\mathbf{x}}\,. \tag{1}$$

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The "braking" force

- The leading order EoM gives rise to $\dot{\vec{\mathbf{a}}} = -\omega_0^2 \vec{\mathbf{v}}$, leading to the radiation reaction force

$$\vec{\mathbf{f}}_{\rm rad} = \frac{e^2}{6\pi\epsilon_0 c^3}\dot{\vec{\mathbf{a}}} = -\frac{e^2\omega_0^2}{6\pi\epsilon_0 c^3}\vec{\mathbf{v}} = -m\Gamma\vec{\mathbf{v}} \,, \tag{2}$$

where $\Gamma = (e^2\omega_0^2)/(6\pi\epsilon_0 mc^3)$.

- The equation of motion, in the absence of any external EM field, is then

$$\ddot{\vec{\mathbf{x}}} + \Gamma\dot{\vec{\mathbf{x}}} + \omega_0^2\vec{\mathbf{x}} = 0 \,. \tag{3}$$

- The $\Gamma\dot{\vec{\mathbf{x}}}$ term is like a general braking/friction term, which may have origins other than the radiation reaction force. The origin of Γ does not matter for the rest of our analysis in this lecture.
- For small Γ, the solution to this equation is

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_0 \exp\left(-i\omega_0 t - \frac{\Gamma t}{2}\right) \,. \tag{4}$$

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The external EM field

- Let the external EM field be a linearly polarized one, with the electric field component $\vec{\mathbf{E}} = E_0 \hat{\mathbf{z}} \exp[i(kx - \omega t)]$.
- If the motion of the electron (equilibrium position $\vec{\mathbf{x}} = \vec{0}$) is limited to a small region ($|\vec{\mathbf{x}}| \ll \lambda$, or $k|\vec{\mathbf{x}}| \ll 1$), the force on this electron will be $\vec{\mathbf{F}}_{\text{ext}} = -eE_0 \hat{\mathbf{z}} \exp(-i\omega t)$.
- The magnetic component of the EM field will also contribute to the Lorentz force on the charge. However, this force is suppressed by a factor v/c compared to eE_0, where v is the typical speed of the electron. Since $v/c \ll 1$ for electrons inside normal materials, we can neglect this component of the force.
- The equation of motion for the electron is then

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Solution for EoM of the electron (1D motion)

- Let us restrict ourselves to a one-dimensional situation, where the motion of the electron is in the same direction as the polarization of $\vec{\mathbf{E}}$, i.e. the z-direction.
- Looking for solutions of the form $z = z_\omega e^{-i\omega t}$, we get

$$-\omega^2 z - i\omega\Gamma z + \omega^2 z = -\frac{e}{m} E_0 \, . \quad (6)$$

The solution is

$$z_\omega = \frac{e}{m} E_0 \frac{1}{(\omega^2 - \omega_0^2) + i\omega\Gamma} \, . \quad (7)$$

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Radiation due to the acceleration of free electron

- For a free electron, there is no harmonic restoring force, so that $\omega_0 = 0$. Let us neglect the radiation reaction force, i.e. we'll work in the limit $\Gamma \to 0$.
- In this situation,

$$z_\omega = \frac{e}{m} E_0 \frac{1}{\omega^2}; \quad , \quad a_\omega = \ddot{z}_\omega = -\frac{e}{m} E_0 \; . \tag{8}$$

Thus, a_ω is independent of ω, the frequency of incident radiation.

- This acceleration leads to the radiation field

$$\vec{\mathbf{E}}_{\rm rad} = \frac{e}{4\pi\epsilon_0 c^2 r^3} [\vec{\mathbf{r}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{a}})] \; . \tag{9}$$

Note that we are in the non-relativistic limit.

- The magnitude of this electric field is

$$|\vec{\mathbf{E}}_{\rm rad}| = \frac{ea \sin\theta}{4\pi\epsilon_0 c^2 r} \tag{10}$$

where θ is the angle made by $\vec{\mathbf{r}}$ with the z axis, which is the direction of acceleration.

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Intensity of radiation

- The magnitude of the Poynting vector, which represents the intensity of radiation, is then

$$I_{\rm rad} = |\vec{\mathbf{N}}_{\rm rad}| = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|\vec{\mathbf{E}}_{\rm rad}|^2 = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\left(\frac{eE_0}{4\pi\epsilon_0 mc^2}\right)^2\frac{\sin^2\theta}{r^2}\,. \quad (11)$$

  where we have used the relation between the acceleration and incident electric field $\vec{\mathbf{E}}$.

- In terms of the intensity of the incident radiation

$$I_0 = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}E_0^2\,, \quad (12)$$

  this may be written as

$$I_{\rm rad} = I_0 \sin^2\theta\frac{r_e^2}{r^2}\,, \quad (13)$$

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Radiated power and Thomson cross section

- The total rate of radiative energy loss is

$$\begin{aligned} \frac{dU}{dt} &= \int I_{\text{rad}} r^2 d\Omega = \int I_0 \sin^2\theta \frac{r_e^2}{r^2} d\Omega \ , \\ &= \frac{8\pi}{3} r_e^2 I_0 \end{aligned} \qquad (14)$$

- This may be interpreted as the scattering of the incident radiation by the electron, resulting in the loss of the incident power by radiation. The scattering cross section is then

$$\sigma_0 = \frac{dU/dt}{I_0} = \frac{8\pi}{3} r_e^2 \qquad (15)$$

  This is the Thomson scattering cross section.

- Note that the Thomson cross section is more than the naive “surface area” of the classical electron, πr_e^2.

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Cross section for scattering off a bound electron

- For a bound electron,

$$z_\omega = \frac{e}{m} E_0 \frac{1}{(\omega^2 - \omega_0^2) + i\omega\Gamma} \,, \quad a_\omega = \ddot{z}_\omega = \frac{e}{m} E_0 \frac{\omega^2}{(\omega^2 - \omega_0^2) + i\omega\Gamma} \,. \tag{16}$$

  Note that now the acceleration depends on ω, the frequency of incident radiation.

- Then the magnitude of the radiated electric field is

$$|\vec{\mathbf{E}}_{\rm rad}| = \frac{qa\sin\theta}{4\pi\epsilon_0 c^2 r} = \frac{q}{4\pi\epsilon_0 mc^2} \frac{\omega^2 E_0}{\sqrt{(\omega^2 - \omega_0^2)^2 + (\omega\Gamma)^2}} \cdot \frac{\sin\theta}{r} \tag{17}$$

- This leads to the scattering cross section

$$\sigma_{\rm scat} = \sigma_0 \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\Gamma)^2} \,. \tag{18}$$

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Resonant scattering

- The scattering cross section $\sigma_{\rm scat}$ as a function of ω becomes large at $\omega = \omega_0$. This is the resonance condition.
- At resonance, the cross section is

$$\sigma_{\rm res} = \sigma_0 \frac{\omega_0^2}{\Gamma^2} \ . \tag{19}$$

For $\Gamma \ll \omega_0$, this could be a large enhancement. The cross section at this frequency can be much larger than the classical electron area.

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Strong binding limit: Rayleigh scattering

- When the restoring harmonic force is much stronger compared to the energy of the incident radiation, i.e. $\omega \ll \omega_0$, one gets

$$\sigma_{\text{large}\,\omega_0} = \sigma_0 \frac{\omega^4}{\omega_0^4} \,, \tag{20}$$

  where we have also assumed $\omega\Gamma \ll \omega_0^2$.

- Thus in the strong binding limit, the scattering cross section increases as the fourth power of the incident frequency. This is Rayleigh scattering.
- Rayleigh scattering implies that the blue colour in the sunlight is scattered much more than the red colour in the atmosphere, and hence the sky looks predominantly blue as long as we are not looking at the sun directly.

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Outline

Work done by the EM field on an electron: 1D

- The EM wave imparts energy to the electron by doing work on it, which can be computed as

$$\Delta U = \mathrm{Re}\left[\int_{-\infty}^{+\infty} e\vec{\mathbf{E}}_0(t)\cdot\dot{\vec{\mathbf{x}}}(t)\,dt\right] = \mathrm{Re}\left[\int_{-\infty}^{+\infty} eE_0(t)\dot{z}(t)\,dt\right], \tag{21}$$

  where the electric field as well as the motion of the electron is along z direction.

- For calculating ΔU, we shall use the Fourier transform identity

$$\int_{-\infty}^{+\infty} p(t)q(t)dt = 2\pi\int_{0}^{+\infty} p_\omega q_\omega d\omega\,, \tag{22}$$

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Work done by the EM field on an electron: 1D

- The EM wave imparts energy to the electron by doing work on it, which can be computed as

$$\Delta U = \mathrm{Re}\left[\int_{-\infty}^{+\infty} e\vec{\mathbf{E}}_0(t) \cdot \dot{\vec{\mathbf{x}}}(t)\, dt\right] = \mathrm{Re}\left[\int_{-\infty}^{+\infty} eE_0(t)\dot{z}(t)\, dt\right], \tag{21}$$

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Energy absorbed by an electron

- The total energy absorbed is

$$\begin{aligned}\Delta U &= 2\pi \frac{e^2}{m} \mathrm{Re}\left[\int_{-\infty}^{+\infty} E_\omega^2 \frac{-i\omega}{(\omega^2 - \omega_0^2) + i\omega\Gamma} d\omega\right] \\ &= 2\pi \frac{e^2}{m} \int_0^{+\infty} E_\omega^2 \frac{2\omega^2\Gamma}{(\omega^2 - \omega_0^2)^2 + \omega^2\Gamma^2} d\omega \end{aligned} \quad (24)$$

  The factor 2 in the integrand comes from the change in the limits of integration.

- For small Γ, the resonance term is sharply peaked and we may approximate

$$\begin{aligned}\Delta U &= 2\pi \frac{e^2}{m} E_{\omega_0}^2 \cdot 2\omega_0^2\Gamma \int_0^{+\infty} \frac{1}{(\omega^2 - \omega_0^2)^2 + \omega^2\Gamma^2} d\omega \\ &= \frac{2\pi e^2}{m} E_{\omega_0}^2 \cdot 2\omega_0^2\Gamma \cdot \frac{\pi}{2\omega_0^2\Gamma} = \frac{2\pi^2 e^2}{m} E_{\omega_0}^2 \end{aligned} \quad (25)$$

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Incident energy flux

- The total incident energy flux per unit area is

$$S = \int_{-\infty}^{+\infty} N(t)dt = \sqrt{\frac{\epsilon_0}{\mu_0}} \int_{-\infty}^{+\infty} E_0^2 dt = \epsilon_0 c \int_{-\infty}^{+\infty} E_0^2 dt \ . \quad (26)$$

- In terms of the Fourier-transformed quantities,

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Absorption cross section

- One may define the absorption cross section at frequency ω as the ratio of total energy absorbed at that frequency to the energy flux per unit area at that frequency.
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$$\Delta U_\omega = 2\pi \frac{e^2}{m} E_\omega^2 \frac{2\omega^2\Gamma}{(\omega^2 - \omega_0^2)^2 + \omega^2\Gamma^2}, \tag{29}$$

  which is just the integrand in the expression for ΔU.
- Then one gets the absorption cross section as

$$\sigma_{\rm abs}(\omega) = \frac{\Delta U_\omega}{S_\omega} = 2\pi \frac{e^2}{m} \frac{2\omega^2\Gamma}{(\omega^2 - \omega_0^2)^2 + \omega^2\Gamma^2} \cdot \frac{1}{4\pi\epsilon_0 c}. \tag{30}$$

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Absorption cross section at resonance

- For small Γ, the absorption takes place only at frequencies close to the resonance frequencies. The incident energy flux at this frequency is $S_{\omega_0} = 4\pi\epsilon_0 c E_{\omega_0}^2$.
- The cross section at this energy is

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Note that the resonant cross section is independent of the actual value of ω_0.

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Outline

Polarization due to an incident EM wave

- At a microscopic level, the refracive index may be viewed as arising from the polarization of the material, which itself is caused by the action of the incident EM field on the electrons.
- In the absence of the external EM field, the material is unpolarized, that is the positive and negative charge densities cancel each other out everywhere inside the material.
- When the EM field is incident on the material, the electrons (free electrons, valence electrons, or electrons which are loosely bound), being lighter, move under its influence. The movement of the rest of the ions may be neglected.
- If the displacement of an individual electron from its mean position is $\vec{\mathbf{x}}$, and the number density of such electrons is N, the collective polarization is

$$\vec{\mathbf{P}} = -N\,e\,\vec{\mathbf{x}}\,. \tag{32}$$

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Wave equation in the presence of polarization

- In the presence of polarization $\vec{\mathbf{P}}$, the Maxwell's equation that gets modified is

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \left(\epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} + \frac{\partial \vec{\mathbf{P}}}{\partial t} \right) \tag{33}$$

- This, combined with $\nabla \times \vec{\mathbf{E}} = -\partial \vec{\mathbf{B}}/\partial t$ yields

$$\begin{aligned} \nabla \times (\nabla \times \vec{\mathbf{E}}) &= -\frac{\partial}{\partial t}(\nabla \times \vec{\mathbf{B}}) \\ \text{i.e.} \quad \nabla(\nabla \cdot \vec{\mathbf{E}}) - \nabla^2 \vec{\mathbf{E}} &= -\left(\mu_0 \epsilon_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} + \mu_0 \frac{\partial^2 \vec{\mathbf{P}}}{\partial t^2} \right) \end{aligned} \tag{34}$$

- Since the net charge density inside the dielectric is zero, $\nabla \cdot \vec{\mathbf{E}} = 0$ and we get

$$\nabla^2 \vec{\mathbf{E}} - \frac{1}{c^2} \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} - \mu_0 \frac{\partial^2 \vec{\mathbf{P}}}{\partial t^2} = 0 \, . \tag{35}$$

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Refractive index

- With the direction of motion of the electrons and the polarization of the incident $\vec{\mathbf{E}}$ field along z direction,

$$\vec{\mathbf{P}} = -N\, e\, z\, \hat{\mathbf{z}} = \frac{Ne^2 E}{m} \frac{1}{(\omega_0^2 - \omega^2) - i\omega\Gamma} \hat{\mathbf{z}}\,. \tag{36}$$

- Thus our situation is that of a linear dielectric, with the polarizability

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Complex refractive index of a dilute electron gas

- Our analysis is valid for dilute electron gases, since we have assumed that the local field $\vec{\mathbf{E}}$ is the same as the incoming field. In such case, we get

$$n^2 = 1 + \chi = 1 + \frac{Ne^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\Gamma} \,. \tag{40}$$

  It is assumed that all the electrons have the same binding frequency ω_0.

- If there are more than one set of electrons with different binding frequencies, the linearity of the problem allows us to just add the contributions to χ.

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Behaviour of the refractive index

- **At low incident frequencies ($\omega < \omega_0$):**
  As long as $\Gamma \ll \omega_0 - \omega$, the refractive index is almost real, and greater than unity. Moreover, when ω increases, the refractive index n also increases. This is the same as the behaviour of light rays through prism.

- **Around $\omega \approx \omega_0$:**
  The refractive index has a significant imaginary component, and the dilute gas acts as an absorber.

- **At large frequencies ($\omega > \omega_0$):**
  The refractive index may be less than 1. Note that this does not lead to faster-than-c propagation. (See Problem on the following page.)

- **At extremely large frequencies ($\omega^2 \gg Ne^2/(m\epsilon_0)$):**
  The refractive index $n \to 1$. (We referred to this while discussing Cherenkov radiation.)

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- Around $\omega \approx \omega_0$:
  The refractive index has a significant imaginary component, and the dilute gas acts as an absorber.

- At large frequencies ($\omega > \omega_0$):
  The refractive index may be less than 1. Note that this does not lead to faster-than-c propagation. (See Problem on the following page.)

- At extremely large frequencies ($\omega^2 \gg Ne^2/(m\epsilon_0)$):
  The refractive index $n \to 1$. (We referred to this while discussing Cherenkov radiation.)

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Dilute gas of free electrons

Problem

Using the expression for the refractive index of a dilute electron gas with number density N, plot the phase velocity and group velocity of an EM wave as a function of ω. Neglect the damping term and choose an appropriate value for N. Where is the “dilute” nature of the gas relevant ?

Problem

Show that a dilute gas of free electrons will not allow an EM wave to propagate through it, if the frequency of the wave is less than a cutoff frequency ω_{cutoff}. Determine the cutoff frequency in terms of the number density of electrons and other universal constants. For $\omega > \omega_{\text{cutoff}}$, qualitatively plot the behaviour of the wavenumber k as a function of ω.

Take-home message from this lecture

- A simple model of electrons bound to the nucleus with a harmonic potential can help model the scattering and absosption of an EM wave
- The cross sections of Thomson scattering, resonance scattering, Rayleigh scattering and resonant absorption follow naturally in this model.
- Essential properties of the refractive index of a dilute electron gas can also be calculated, and its frequency behaviour studied.