Module I: Electromagnetic waves

Lecture 3: EM waves with boundaries, confined spaces

Amol Dighe
TIFR, Mumbai

Outline

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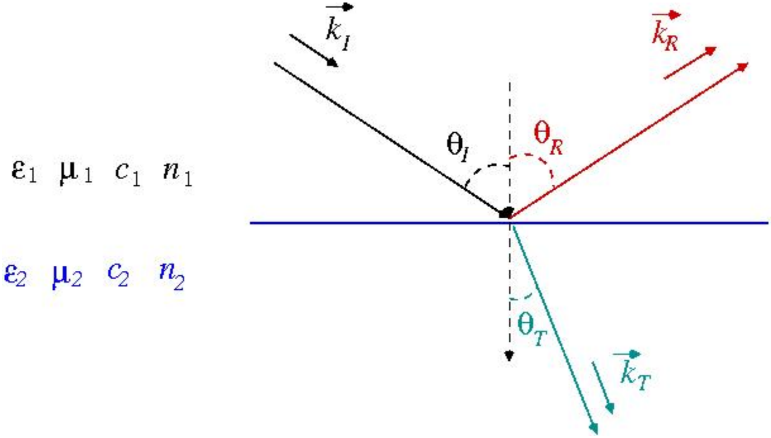
Reflection and refraction

An EM wave is incident from one medium ($\epsilon_1, \mu_1, n_1, c_1$) to another medium ($\epsilon_1, \mu_1, n_1, c_1$), at an angle θ_I with the normal to the boundary.



Incident, reflected and refracted waves

Incident wave

$$\vec{\mathbf{E}}_I = \vec{\mathbf{E}}_{I0} e^{i(\vec{\mathbf{k}}_I \cdot \vec{\mathbf{r}} - \omega t) + i\phi_I} \tag{1}$$

$$\vec{\mathbf{B}}_I = \frac{\vec{\mathbf{k}}_I}{\omega} \times \vec{\mathbf{E}}_I = \frac{1}{c_1}(\hat{\mathbf{k}}_I \times \vec{\mathbf{E}}_I) \tag{2}$$

Reflected wave

$$\vec{\mathbf{E}}_R = \vec{\mathbf{E}}_{R0} e^{i(\vec{\mathbf{k}}_R \cdot \vec{\mathbf{r}} - \omega t) + i\phi_R} \tag{3}$$

$$\vec{\mathbf{B}}_R = \frac{\vec{\mathbf{k}}_R}{\omega} \times \vec{\mathbf{E}}_R = \frac{1}{c_1}(\hat{\mathbf{k}}_R \times \vec{\mathbf{E}}_R) \tag{4}$$

Transmitted wave

$$\vec{\mathbf{E}}_T = \vec{\mathbf{E}}_{T0} e^{i(\vec{\mathbf{k}}_T \cdot \vec{\mathbf{r}} - \omega t) + i\phi_T} \tag{5}$$

$$\vec{\mathbf{B}}_T = \frac{\vec{\mathbf{k}}_T}{\omega} \times \vec{\mathbf{E}}_T = \frac{1}{c_2}(\hat{\mathbf{k}}_T \times \vec{\mathbf{E}}_T) \tag{6}$$

Boundary conditions on phases

$\vec{\mathbf{D}}_\perp$ is continuous across the boundary

$$\epsilon_1 \vec{\mathbf{E}}_{I\perp} + \epsilon_1 \vec{\mathbf{E}}_{R\perp} = \epsilon_2 \vec{\mathbf{E}}_{T\perp} \quad (7)$$

$$\epsilon_1 \vec{\mathbf{E}}_{I\perp 0} e^{i(\vec{\mathbf{k}}_I \cdot \vec{\mathbf{r}} - \omega t) + i\phi_I} + \epsilon_1 \vec{\mathbf{E}}_{R\perp 0} e^{i(\vec{\mathbf{k}}_R \cdot \vec{\mathbf{r}} - \omega t) + i\phi_R} = \epsilon_2 \vec{\mathbf{E}}_{T\perp 0} e^{i(\vec{\mathbf{k}}_T \cdot \vec{\mathbf{r}} - \omega t) + i\phi_T} \quad (8)$$

- The equality should be valid at all $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2$ on the boundary

$$\vec{\mathbf{k}}_I \cdot (\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) = \vec{\mathbf{k}}_R \cdot (\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) = \vec{\mathbf{k}}_T \cdot (\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) \quad (9)$$

- Taking the origin at $\vec{\mathbf{r}}_2$ (any point on the boundary):

$$|\vec{\mathbf{k}}_I| r \sin\theta_I = |\vec{\mathbf{k}}_R| r \sin\theta_R = |\vec{\mathbf{k}}_T| r \sin\theta_T \quad (10)$$

- Using $|k_I| = |k_R|$ and $|k_T|/|k_I| = n_2/n_1$,

$$\sin\theta_I = \sin\theta_R\ , \quad \frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1} = \frac{c_1}{c_2} \quad (11)$$

The first is the law of reflection the second is the Snell's law

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Boundary conditions on amplitudes

The discussion on the previous page would have worked for any of the boundary conditions, we just took $\vec{\mathbf{D}}_\perp$ as an example. Now we need not worry about the phases, since the laws of reflection and refraction derived there guarantee that the phase conditions will be satisfied.

Boundary conditions

$$\epsilon_1 \vec{\mathbf{E}}_{I\perp 0} + \epsilon_1 \vec{\mathbf{E}}_{R\perp 0} = \epsilon_2 \vec{\mathbf{E}}_{T\perp 0} \tag{12}$$

$$\vec{\mathbf{B}}_{I\perp 0} + \vec{\mathbf{B}}_{R\perp 0} = \vec{\mathbf{B}}_{T\perp 0} \tag{13}$$

$$\vec{\mathbf{E}}_{I||0} + \vec{\mathbf{E}}_{R||0} = \vec{\mathbf{E}}_{T||0} \tag{14}$$

$$\frac{1}{\mu_1}\vec{\mathbf{B}}_{I||0} + \frac{1}{\mu_1}\vec{\mathbf{B}}_{R||0} = \frac{1}{\mu_2}\vec{\mathbf{B}}_{T||0} \tag{15}$$

For convenience we'll divide the incident electric field into a component in the plane of incidence (the plane that contains $\vec{\mathbf{k}}_I, \vec{\mathbf{k}}_R, \vec{\mathbf{k}}_T$) and a component normal to the plane of incidence. These two clearly won't interfere, and they can be added together at any time, using the principle of superposition, to get the net electric field.

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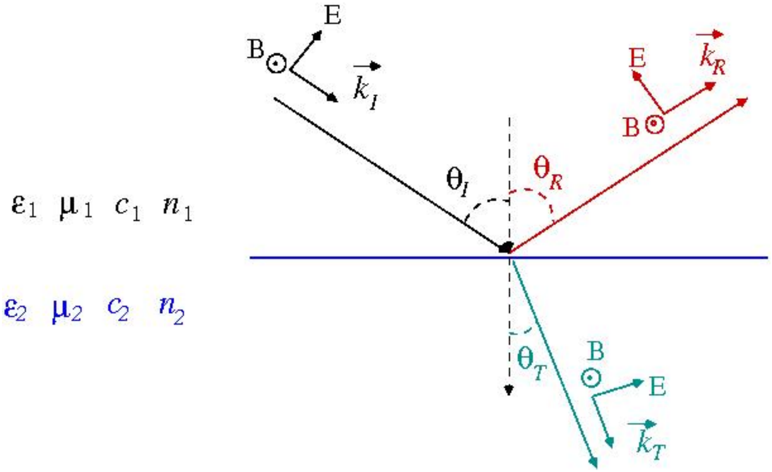

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$\vec{k}_I$
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B
θ_T
E
$\vec{k}_T$

Applying boundary conditions

Boundary conditions involving $\vec{\mathbf{E}}$

$$-\epsilon_1 E_{I0} \sin\theta_I + \epsilon_1 E_{R0} \sin\theta_R = -\epsilon_2 E_{T0} \sin\theta_T \quad (16)$$

$$-E_{I0} \cos\theta_I + E_{R0} \cos\theta_R = E_{T0} \cos\theta_T \quad (17)$$

- Solution:

$$E_{R0} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) E_{I0}\,, \quad E_{T0} = \left(\frac{2}{\alpha + \beta}\right) E_{I0} \quad (18)$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}\,, \quad \beta = \frac{\epsilon_2}{\epsilon_1}\frac{c_2}{c_1} \quad (19)$$

Boundary conditions involving $\vec{\mathbf{B}}$ give exactly the same conditions.

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Reflection and transmission coefficient

- Rate of energy transported by incoming wave normal to the boundary:

$$\text{Incident wave}: \quad I_I = \frac{1}{2}\epsilon_1 c_1 |\vec{\mathbf{E}}_{I0}|^2 \, |\cos\theta_I| \tag{20}$$

$$\text{Reflected wave}: \quad I_R = \frac{1}{2}\epsilon_1 c_1 |\vec{\mathbf{E}}_{R0}|^2 \, |\cos\theta_R| \tag{21}$$

$$\text{Transmitted wave}: \quad I_T = \frac{1}{2}\epsilon_2 c_2 |\vec{\mathbf{E}}_{T0}|^2 \, |\cos\theta_T| \tag{22}$$

- Reflection coefficient

$$R = \frac{I_R}{I_I} = \left|\frac{\alpha - \beta}{\alpha + \beta}\right|^2 \tag{23}$$

- Transmission coefficient

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 c_2}{\epsilon_1 c_1}\frac{|\cos\theta_T|}{|\cos\theta_I|} = \frac{4\mathrm{Re}(\alpha\beta)}{|\alpha+\beta|^2} \tag{24}$$

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Comments on reflection and transmission coefficients

- $R + T = 1$, as expected
- $R = 1, T = 0$ possible if α is purely imaginary.

$$\alpha = \frac{\sqrt{1 - \text{“}\sin^2\theta_T\text{''}}}{\cos\theta_I} = \frac{\sqrt{1 - (n_2/n_1)^2 \sin^2\theta_I}}{\cos\theta_I}, \qquad (25)$$

so if $\sin\theta_I > (n_1/n_2)$, there is no transmission.
This is the condition for Total Internal reflection.

- $R = 0, T = 1$ possible if $\alpha = \beta$. This condition takes a simple form if $\mu_1 = \mu_2$, since then

$$\frac{\cos\theta_T}{\cos\theta_I} = \frac{\epsilon_2 c_2}{\epsilon_1 c_1} = \frac{c_1}{c_2}; \quad \text{and} \quad \frac{\sin\theta_I}{\sin\theta_T} = \frac{c_1}{c_2} \qquad (26)$$

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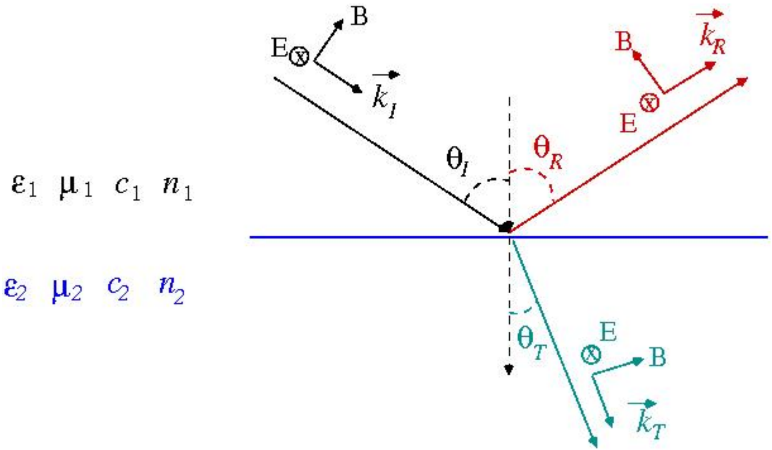

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Comments on this scenario

- The values for R and T will in general be different. In particular, $R = 0$ is not possible here.
- If an unpolarized wave is incident on a dielectric surface, the reflected and transmitted waves will therefore, in general, be polarized.
- $\Rightarrow$ Homework !

Outline

Reflection from a conducting surface

- No wave is transmitted inside the conductor; i.e. fields inside the conductor are zero.
- For incidence at an angle, the components of $\vec{\mathbf{E}}_I$ and $\vec{\mathbf{E}}_R$ parallel to the boundary cancel, i.e. $\vec{\mathbf{E}}_{I\parallel} = -\vec{\mathbf{E}}_{R\parallel}$
  (For normal incidence, $\vec{\mathbf{E}}_I = -\vec{\mathbf{E}}_R$.)
- There will be charge oscillations at the metal surface corresponding to $\epsilon_1(\vec{\mathbf{E}}_{I\perp} + \vec{\mathbf{E}}_{R\perp}) = \sigma_s$, where σ_s is the surface charge density
- The movements of these charges along the surface correspond to surface currents, which account for finite values of $\vec{\mathbf{H}}_{I\parallel} + \vec{\mathbf{H}}_{R\parallel}$ at the boundary.
- The net $\vec{\mathbf{B}}$ normal to the surface vanishes, i.e. $\vec{\mathbf{B}}_{I\perp} + \vec{\mathbf{B}}_{R\perp} = 0$. This follows automatically from the $\vec{\mathbf{E}}_{\parallel}$ conditions above.

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- There will be charge oscillations at the metal surface corresponding to $\epsilon_1(\vec{\mathbf{E}}_{I\perp} + \vec{\mathbf{E}}_{R\perp}) = \sigma_s$, where σ_s is the surface charge density
- The movements of these charges along the surface correspond to surface currents, which account for finite values of $\vec{\mathbf{H}}_{I\parallel} + \vec{\mathbf{H}}_{R\parallel}$ at the boundary.
- The net $\vec{\mathbf{B}}$ normal to the surface vanishes, i.e. $\vec{\mathbf{B}}_{I\perp} + \vec{\mathbf{B}}_{R\perp} = 0$. This follows automatically from the $\vec{\mathbf{E}}_\parallel$ conditions above.

Recap of topics covered so far

- Reflection and transmission at the surface of a dielectric
- Boundary conditions at a conducting surface

Outline

Travelling waves with the same (x, y) profile

- We are looking for waves travelling in z direction, while keeping the same (x, y) profile. I.e. the form

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}^0(x, y)e^{i(k_z z - \omega t)} , \quad \vec{\mathbf{B}} = \vec{\mathbf{B}}^0(x, y)e^{i(k_z z - \omega t)} \tag{27}$$

- Maxwell's $(\nabla \times \vec{\mathbf{E}})$ and $(\nabla \times \vec{\mathbf{B}})$ equations then become

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z , \qquad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2} E_z \tag{28}$$

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- Note that one can factor out the $e^{i(k_z z - \omega t)}$ dependence of E_x, E_y, E_z and B_x, B_y, B_z, so now onwards they have no z- or t-dependence in this lecture.

- Using the last two lines (4 equations), one can write E_x, E_y, B_x, B_y in terms of the other two quantities, E_z and B_z

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All components in terms of E_z and B_z

$$E_x = \frac{1}{(\omega/c)^2 - k_z^2}\left(k_z\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right) \quad (31)$$

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Constraining E_z, B_z themselves

E_z, B_z themselves must satisfy consistency conditions

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These correspond to

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EM wave propagation in waveguides

- Let us consider rectangular / circular hollow conducting cylinders, through which an EM wave will be "guided" by bending the boundaries of the cylinders.
- A simple solution would have been a plane wave travelling along z direction, such that $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are transverse, $E_z = B_z = 0$. Such a solution is called as TEM (transverse electric and magnetic) mode.
- Such a mode is not possible in a hollow cylinder, proof given on the next page
- However E_z and B_z can individually vanish, such modes are termed TE (Transverse electric: $E_z = 0$) and TM (Transverse magnetic: $B_z = 0$).

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Hollow cylinder cannot have both $E_z = 0$ and $B_z = 0$

- Lines of $\vec{\mathbf{B}}$ field must form closed loops in the $x - y$ plane.
- Since $\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}_{\text{ext}} + \mu_0 \epsilon_0 \partial \vec{\mathbf{E}} / \partial t$, there must be a current along the axis of the cylinder (z direction), either as $\vec{\mathbf{J}}_{\text{ext}}$ or as $\partial \vec{\mathbf{E}} / \partial t$.
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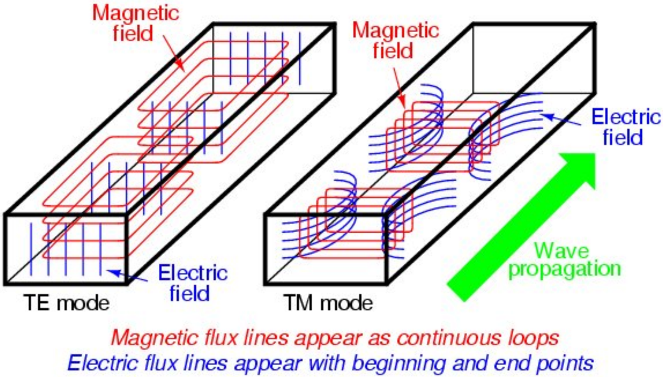
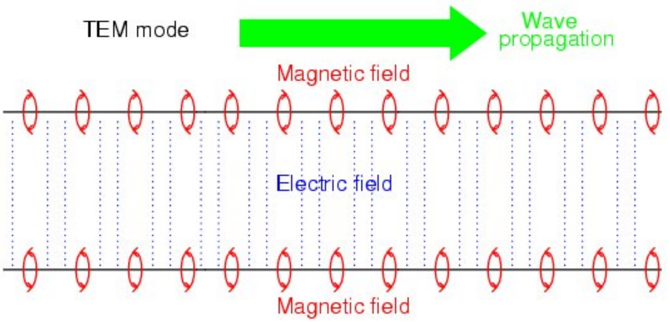
TE and TM modes



TEM mode



Both field planes perpendicular (transverse) to direction of signal propagation.

Outline

TE modes ($E_z = 0, B_z \neq 0$) in a rectangular waveguide

- Let the walls of the waveguide be at $y = 0, b$ and $x = 0, a$. The boundary conditions are then
  $E_x = 0$ at $y = 0, b$ and $E_y = 0$ at $x = 0, a$
- The equations that give $E_{x,y}$ in terms of B_z then imply
  $\frac{\partial B_z}{\partial y} = 0$ at $y = 0, b$ and $\frac{\partial B_z}{\partial x} = 0$ at $x = 0, a$
- The solution to the differential equation for B_z, with these boundary conditions, is

$$B_z = A\cos(k_x x)\cos(k_y y) \tag{39}$$

  where $k_x = (m\pi/a)$ and $k_y = (n\pi/b)$.
- Such a mode is called TE_{mn} mode. Note that at least one of m or n has to be nonzero, else all fields will vanish.

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Cutoff frequencies for TE modes

- The TE_{mn} solution, when substituted in the differential equation for B_z, gives

$$-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 - k_z^2 + (\omega/c)^2 = 0 \tag{40}$$

- For consistency with the physical situation, k_z must be real; i.e. $k_z^2 > 0$. This gives the condition

$$\omega > c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \equiv \omega_{mn} \tag{41}$$

- Thus, for a TE mode TE_{mn} to propagate, it must have a minimum frequency ω_{mn}. A waveguide thus acts like a high-pass filter.

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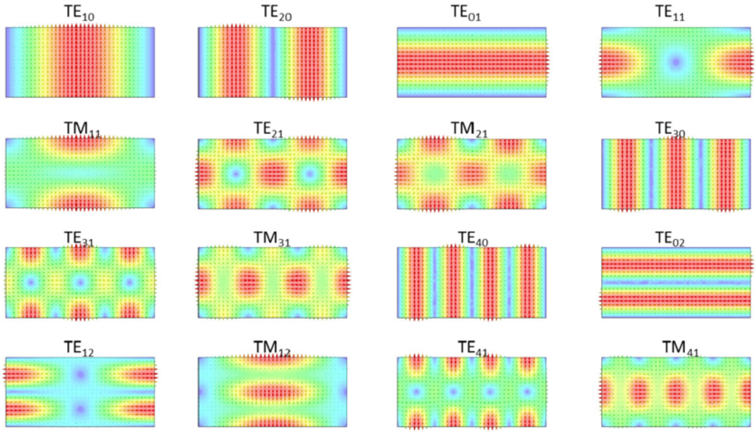
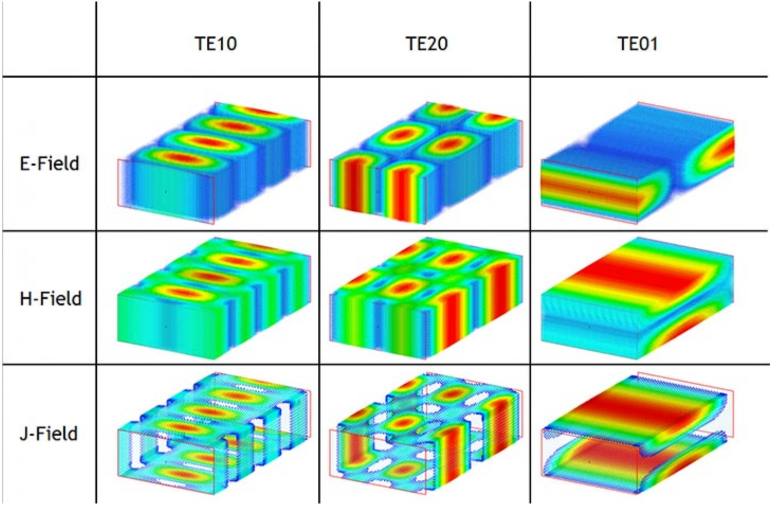
TM modes

- A similar analysis is possible for TM modes, but this will not be done here.
- Note that the cutoff frequencies ω_{mn} for the TM modes are the same as those for TE modes.

Electric fields in TE and TM mode



Electric field, magnetic field, current



Phase velocity and group velocity

- Phase velocity: simply the speed at which the crest of the wavefront travels in a given direction.
- For a plane wave $Ae^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}-\omega t)}$, the phase velocity along the direction $\hat{\mathbf{r}}$ is

$$v_{ph} = \left.\frac{dr}{dt}\right|_{\text{constant phase}} = \frac{\omega}{|\vec{\mathbf{k}}.\hat{\mathbf{r}}|} \tag{42}$$

- If $\vec{\mathbf{k}}$ is not along $\hat{\mathbf{r}}$, typically $v_{ph} > c$. This does not mean that any signal is travelling faster than light, though.
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Phase velocity and group velocity

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Group velocity

- The Fourier transform of a wave gives the frequencies the wave consists of. Consider the situation where the spread in frequencies is small, which is the only one where we can define a group velocity easily. Let the frequencies be confined to the range $\omega = \omega_0 \pm \Delta\omega$. The corresponding wave vectors are confined to $\vec{\mathbf{k}} = \vec{\mathbf{k}}_0 \pm \Delta\vec{\mathbf{k}}$.
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$$\psi(\vec{\mathbf{x}}, t) = \int a(\vec{\mathbf{k}}) e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}} - \omega t)} d^3k \ , \tag{43}$$
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Group velocity: continued

- Let us consider a one-dimensional case of a wave travelling along z-axis. At the peak,

$$0 = \frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial z}\frac{dz}{dt} \tag{46}$$

- The group velocity is then

$$v_g = \frac{dz}{dt} = -\frac{\partial A/\partial t}{\partial A/\partial z} = \frac{\Delta\omega}{\Delta k} = \left.\frac{d\omega}{dk}\right|_{\omega_0} \tag{47}$$

Velocities along z axis for the waveguide

- $\omega = \sqrt{\omega_{mn}^2 + k_z^2}$
- Phase velocity $v_{ph} = \frac{\omega}{k_z} = \frac{c}{\sqrt{1-(\omega_{mn}/\omega)^2}}$
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- Waveguide transports different frequencies at different speeds: dispersion

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Outline

Circular cylindrical waveguides

No TEM mode, as per the earlier arguments

For TM mode

$$E_z = AJ_m(k_\ell r)e^{im\phi}e^{i(k_z z - \omega t)} \tag{48}$$

If the cylinder has radius r_0,
then the boundary condition is $J_m(k_\ell r_0) = 0$, gives $k_\ell(m)$

For TE mode

$$B_z = AJ_m(k_\ell r)e^{im\phi}e^{i(k_z z - \omega t)} \tag{49}$$

If the cylinder has radius r_0,
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- $k_z^2 = (\omega/c)^2 - k_\ell^2 \Rightarrow$ cutoff frequency $\omega_{m,\ell} = ck_\ell(m)$
- TM and TE modes have different cutoff frequencies, unlike rectangular waveguides !

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Power transmitted by a waveguide

- Consider TE mode. i.e. $E_z = 0$.
- The equations for $\vec{\mathbf{E}}_\perp = (E_x, E_y)$ and $\vec{\mathbf{B}}_\perp = (B_x, B_y)$ become

$$\left.\begin{aligned} B_x &= \frac{ik_z}{k_\perp^2}\frac{\partial B_z}{\partial x} \\ B_y &= \frac{ik_z}{k_\perp^2}\frac{\partial B_z}{\partial y} \end{aligned}\right\} \Rightarrow \quad \vec{\mathbf{B}}_\perp = \frac{ik_z}{k_\perp^2}\nabla_\perp B_z \tag{50}$$

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- Comparing with $|\vec{\mathbf{N}}| = (1/2)\sigma|E_0|^2$, this enables us to define the conductance of the waveguide as $\sigma = \sqrt{\epsilon_0/\mu_0}(k_z/k)$. This may be compared with the conductance of free space, $\sqrt{\epsilon_0/\mu_0}$.

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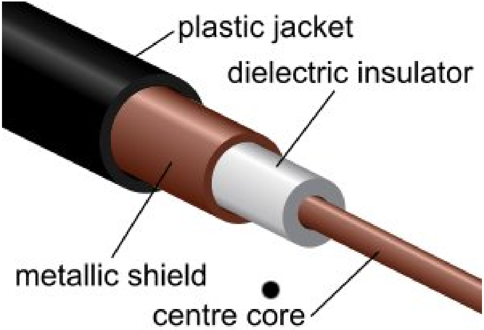
Outline

1. EM waves at dielectric boundaries: reflection, transmission
   - $\vec{E}$ in the plane of incidence
   - $\vec{E}$ normal to the plane of incidence
2. EM waves in conductors: inside and at the boundary
3. Waveguides
   - Rectangular waveguide
   - Circular cylindrical waveguides
4. **Coaxial cable and cavities**

Coaxial cable and cavities



Propagation through a coaxial cable

- TEM Mode is supported (now there are two disjoint boundaries, so the argument for hollow waveguides does not work.)
- TE and TM modes also propagate, but have a threshold frequency

The TEM mode

- Electric and magnetic fields:

$$\vec{\mathbf{E}} = \frac{E_0 \hat{\mathbf{r}}}{r} e^{i(k_z z - \omega t)} \;, \quad \vec{\mathbf{B}} = \frac{E_0 \hat{\phi}}{cr} e^{i(k_z z - \omega t)} \tag{53}$$

- Group velocity $v_g = c$

Rectangular cavity

- Conducting walls at $x = 0, a$; at $y = 0, b$ and at $z = 0, c$.
- Potential inside the cavity:

$$\Phi_{mnp} = \sin(k_x x)\sin(k_y y)\sin(k_z z)e^{-i\omega t} \tag{54}$$

where $k_x = (m\pi/a), k_y = (n\pi/b), k_z = (p\pi/c)$

- This can be used to obtain $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ inside the cavity.
- A rectangular cavity supports discrete modes.

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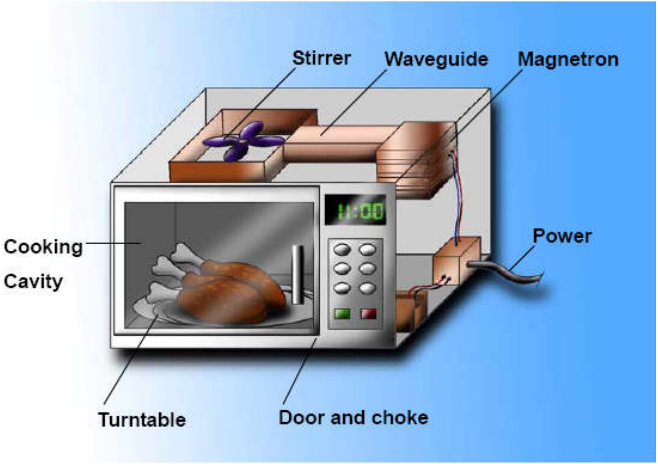
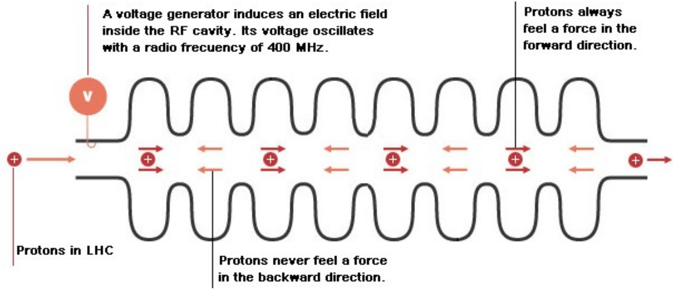
Microwave: waveguide and cavity



LHC accelerator: cavity principle



LHC accelerator: bunching cavities

Recap of topics in waveguides and cavities

- Propagation in waveguides in terms of E_z and B_z
- TEM, TE and TM modes from Maxwell's equations
- No TEM modes for hollow waveguides
- Waveguides as high-pass filters, as dispersive media
- Phase velocity and group velocity
- Power transmitted through waveguides
- Coaxial cable: TEM propagation, in addition to TE and TM
- Cavities for bunching protons together at accelerators