Module I: Electromagnetic waves

Lecture 5: EM wave equation with sources

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Outline

1. Wave equation for ϕ and $\vec{A}$ with sources

2. Solving the wave equation with sources

Outline

Why the potentials $\vec{\mathbf{A}}$ and ϕ ?

- We already have the wave equation satisfied by $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ in the absence of any charge or current sources:

$$\nabla^2\vec{\mathbf{E}} - \mu\sigma\frac{\partial\vec{\mathbf{E}}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} = 0 \quad (1)$$

  and similarly for $\vec{\mathbf{B}}$.

- However when sources (ρ and $\vec{\mathbf{J}}$) are introduced, they affect $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ in rather complicated ways. Therefore (with hindsight) we formulate our problem in terms of $\vec{\mathbf{A}}$ and ϕ, the vector and scalar potentials, respectively.
- When we come to relativity and covariance of equations, we'll appreciate the importance of $\vec{\mathbf{A}}$ and ϕ even more.

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A and ϕ: definitions

- Maxwell's equation $\nabla \cdot \vec{\mathbf{B}} = 0$ allows us to write $\vec{\mathbf{B}}$ as the curl of a vector, we define this vector as $\vec{\mathbf{A}}$:

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}} \tag{2}$$

  Note that this does not define $\vec{\mathbf{A}}$ completely, since $\nabla \cdot \vec{\mathbf{A}}$ has not yet been defined, so the uniqueness theorem is not satisfied.

- Maxwell's equation $\nabla \times \vec{\mathbf{E}} = -\partial \vec{\mathbf{B}}/\partial t$ then implies

$$\nabla \times \vec{\mathbf{E}} = -\nabla \times \frac{\partial \vec{\mathbf{A}}}{\partial t} \tag{3}$$

  This allows us to write

$$\vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{A}}}{\partial t} - \nabla \phi \tag{4}$$

  where ϕ is a scalar. This is the definition of ϕ.

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Gauge freedom for $\vec{\mathbf{A}}$ and ϕ

- As obserbed earlier, A and ϕ are not uniquely defined.
- Indeed, we can carry out simultaneous gauge trasformations

$$\vec{\mathbf{A}}' = \vec{\mathbf{A}} - \nabla\psi \, , \phi' = \phi + \frac{\partial\psi}{\partial t} \tag{5}$$

  with any arbitrary scalar ψ, and these new potentials $\vec{\mathbf{A}}'$ and ϕ' will still give us the same $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$.
- Since $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are the physically measurable quantities, the potentials $(\vec{\mathbf{A}}, \phi)$ and $(\vec{\mathbf{A}}', \phi')$ are equivalent
- This freedom of choosing any ψ corresponds to the "gauge symmetry". We can choose to do the calculations in any convenient gauge, the final measurable quantities will turn out to be identical / gauge invariant.

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Wave equation for $\vec{\mathbf{A}}$

- We have already used two Maxwell's equations while defining $\vec{\mathbf{A}}$ and ϕ: they will be satisfied automatically.
- Using $\nabla \times \vec{\mathbf{B}} = \mu \vec{\mathbf{J}} + \mu\epsilon(\partial \vec{\mathbf{E}}/\partial t)$, where $\vec{\mathbf{J}} = \vec{\mathbf{J}}_{\rm ext} + \sigma \vec{\mathbf{E}}$, we get

$$\nabla \times (\nabla \times \vec{\mathbf{A}}) = \mu \vec{\mathbf{J}}_{\rm ext} + \mu\sigma \left(-\frac{\partial \vec{\mathbf{A}}}{\partial t} - \nabla\phi \right) + \epsilon\mu \left(-\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} - \nabla \frac{\partial \phi}{\partial t} \right) \tag{6}$$

- Using $\nabla \times (\nabla \times \vec{\mathbf{A}}) = -\nabla^2 \vec{\mathbf{A}} + \nabla(\nabla \cdot \vec{\mathbf{A}})$, this leads to

$$\nabla^2 \vec{\mathbf{A}} - \mu\sigma \frac{\partial \vec{\mathbf{A}}}{\partial t} - \epsilon\mu \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu \vec{\mathbf{J}}_{\rm ext} + \nabla(\nabla \cdot \vec{\mathbf{A}}) + \nabla(\mu\sigma\phi) + \nabla \left(\epsilon\mu \frac{\partial \phi}{\partial t} \right) \tag{7}$$

- If we now use our gauge freedom to make $\nabla \cdot \vec{\mathbf{A}} + \mu\sigma\phi + \epsilon\mu(\partial\phi/\partial t) = 0$, (called as the Lorenz gauge), the we get the wave equation for $\vec{\mathbf{A}}$:

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$$\nabla \cdot \left(-\frac{\partial \vec{\mathbf{A}}}{\partial t} - \nabla \phi \right) = \frac{\rho}{\epsilon} \Rightarrow -\frac{\partial}{\partial t}(\nabla \cdot \vec{\mathbf{A}}) - \nabla^2 \phi = \frac{\rho}{\epsilon} \qquad (9)$$

- Now we use the same Lorenz condition as before to replace $\nabla \cdot \vec{\mathbf{A}}$ by $-\mu\sigma\phi - \epsilon\mu(\partial\phi/\partial t)$, which leads to

$$\nabla^2 \phi - \mu\sigma \frac{\partial \phi}{\partial t} - \epsilon\mu \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon} \qquad (10)$$

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The wave equations in vacuum

- In vacuum, the wave equations take the form

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{11}$$

$$\nabla^2\vec{\mathbf{A}} - \frac{1}{c^2}\frac{\partial^2\vec{\mathbf{A}}}{\partial t^2} = -\mu_0\vec{\mathbf{J}} \tag{12}$$

- We already know that for the static situation, i.e. when the $(\partial^2/\partial t^2)$ terms are absent:

$$\nabla^2\phi(\vec{\mathbf{x}}) = -\frac{\rho(\vec{\mathbf{x}})}{\epsilon_0} \Rightarrow \phi(\vec{\mathbf{x}}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{\mathbf{x}}')}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}d^3x' \tag{13}$$

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Fourier analysis

- Let us try solving a general equation

$$\nabla^2\psi(\vec{\mathbf{x}},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\vec{\mathbf{x}},t) = -g(\vec{\mathbf{x}},t) \qquad (15)$$

by using the method of Fourier transform and Green's function.

- Write the solution $\psi(\vec{\mathbf{x}},t)$ and the source $g(\vec{\mathbf{x}},t)$ in terms of their Fourier transforms ψ_ω and g_ω:

$$\psi(\vec{\mathbf{x}},t) = \int_{-\infty}^{\infty}\psi_\omega(\vec{\mathbf{x}})e^{-i\omega t}d\omega\ ,\ \ g(\vec{\mathbf{x}},t) = \int_{-\infty}^{\infty}g_\omega(\vec{\mathbf{x}})e^{-i\omega t}d\omega\ , \qquad (16)$$

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- In terms of the Fourier transforms, the wave equation becomes

$$\nabla^2\psi_\omega(\vec{\mathbf{x}}) + \frac{\omega^2}{c^2}\psi_\omega(\vec{\mathbf{x}}) = -g_\omega(\vec{\mathbf{x}}) \qquad (18)$$

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The method of Green's function

- The method of Green's functions implies that:
  If $G(\vec{\mathbf{x}}, \vec{\mathbf{x}}')$ is a solution to the Green's equation

$$\nabla^2 G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') + \frac{\omega^2}{c^2} G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = -\delta(\vec{\mathbf{x}} - \vec{\mathbf{x}}') \tag{19}$$

  then the solution to $\psi_\omega(\vec{\mathbf{x}})$ is obtained as

$$\psi_\omega(\vec{\mathbf{x}}) = \int g_\omega(\vec{\mathbf{x}}') G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') d^3x' \tag{20}$$

  This may be checked by explicit substitution.
- The Green's equation is spherically symmetric, so we expect a spherically symmetric solution, i.e. $G(\vec{\mathbf{x}} - \vec{\mathbf{x}}')$ is simply $G(r)$.
- The Green's equation is then

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}[rG(r)] + k^2 G(r) = -\delta(r) \tag{21}$$

- This has a solution (that may be checked by substitution):

$$G(r) = \frac{1}{4\pi r} e^{\pm ikr} \Rightarrow G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = \frac{1}{4\pi|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} e^{\pm ik|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} \tag{22}$$

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  then the solution to $\psi_\omega(\vec{\mathbf{x}})$ is obtained as

$$\psi_\omega(\vec{\mathbf{x}}) = \int g_\omega(\vec{\mathbf{x}}') G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') d^3x' \tag{20}$$

  This may be checked by explicit substitution.
- The Green's equation is spherically symmetric, so we expect a spherically symmetric solution, i.e. $G(\vec{\mathbf{x}} - \vec{\mathbf{x}}')$ is simply $G(r)$.
- The Green's equation is then

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}[rG(r)] + k^2 G(r) = -\delta(r) \tag{21}$$

- This has a solution (that may be checked by substitution):

$$G(r) = \frac{1}{4\pi r} e^{\pm ikr} \Rightarrow G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = \frac{1}{4\pi|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} e^{\pm ik|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} \tag{22}$$

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Solution for $\psi_\omega(\vec{\mathbf{x}})$ and $\psi(\vec{\mathbf{x}}, t)$

- The Green's function method has now given us the solution for $\psi_\omega(\vec{\mathbf{x}})$:

$$\psi_\omega(\vec{\mathbf{x}}) = \frac{1}{4\pi} \int \frac{g_\omega(\vec{\mathbf{x}}')}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} e^{\pm ik|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{23}$$

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- In terms of $t_\pm \equiv t \pm |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|/c$, this becomes

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Properties of the solution to wave equation

- The solution formally looks the same as the solution for the static case, except the time dependence, which appears through t_+ and t_-: the advanced and retarded times respectively.
- This implies that the potentials at any point depend on the source distribution at some other times: in particular, at times $t_\pm = t \pm |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|/c$. This is akin to a signal taking time $|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|/c$ to travel from the source at $\vec{\mathbf{x}}'$ to affect the potential at x.
- Thus, the disturbance caused by the sources travels with the speed c. That is, the speed of light is c.
- When we are dealing with the effect of time-varying sources on the potentials, advanced solutions are not physical since they would violate causality.

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The retarded potentials

- The retarded potentials, caused by time-varying sources, are:

$$\phi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\vec{\mathbf{x}}', t - \frac{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}{c}\right)}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' = \frac{1}{4\pi\epsilon_0} \int \frac{[\rho(\vec{\mathbf{x}}')]}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{26}$$

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{J}}\left(\vec{\mathbf{x}}', t - \frac{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}{c}\right)}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' = \frac{\mu_0}{4\pi} \int \frac{[\vec{\mathbf{J}}(\vec{\mathbf{x}}')]}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{27}$$

where $[f(\vec{\mathbf{x}})]$ is a convention used to write $f\left(\vec{\mathbf{x}}, t - \frac{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}{c}\right)$

Recap of topics covered in this lecture

- Definitions of the potentials $\vec{\mathbf{A}}$ and ϕ, gauge freedom
- Lorenz gauge and wave equations for $\vec{\mathbf{A}}$ and ϕ in the presence of sources (charges and currents)
- Solution to the wave equation in vacuum, using Fourier transforms and Green's function
- Advanced and retarded solutions for the potentials $\vec{\mathbf{A}}$ and ϕ