Module I: Electromagnetic waves

Lecture 6: EM radiation

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Outline

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1. Electric and magnetic fields: radiation components
2. Monochromatic sources
3. Energy carried by radiation

$\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ from $\vec{\mathbf{A}}$ and ϕ

- Now that we know

$$\phi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{\mathbf{x}}', t_-)}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \ , \quad \vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{J}}(\vec{\mathbf{x}}', t_-)}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{1}$$

- We can calculate $\vec{\mathbf{E}} = -\nabla\phi - \partial\vec{\mathbf{A}}/\partial t$ and $\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}}$ to obtain $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$. However note the subtle point:

$$\nabla = \hat{\mathbf{x}}^i \left.\frac{\partial}{\partial x^i}\right|_t \ , \quad \frac{\partial}{\partial t} = \left.\frac{\partial}{\partial t}\right|_{\vec{\mathbf{x}}} \ , \tag{2}$$

  whereas $\rho(\vec{\mathbf{x}}', t_-)$ and $\vec{\mathbf{J}}(\vec{\mathbf{x}}', t_-)$ are specified in terms of the retarded time t_-.

- Hence, we'll have to use

$$\begin{aligned} \left.\frac{\partial\rho(\vec{\mathbf{x}}', t_-)}{\partial x^i}\right|_t &= \frac{\partial\rho(\vec{\mathbf{x}}', t_-)}{\partial t_-} \cdot \left.\frac{\partial t_-}{\partial x^i}\right|_t \\ \left.\frac{\partial\vec{\mathbf{J}}(\vec{\mathbf{x}}', t_-)}{\partial x^i}\right|_t &= \frac{\partial\vec{\mathbf{J}}(\vec{\mathbf{x}}', t_-)}{\partial t_-} \cdot \left.\frac{\partial t_-}{\partial x^i}\right|_t \end{aligned} \tag{3}$$

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Intermediate derivatives: gradient of ϕ

- Defining $\vec{\mathbf{r}} \equiv \vec{\mathbf{x}} - \vec{\mathbf{x}}' = (r_1, r_2, r_3)$ and $r \equiv |\vec{\mathbf{r}}|$, we have

$$\frac{\partial}{\partial x^i} \frac{\rho(\vec{\mathbf{x}}', t_-)}{r}\bigg|_t = \rho(\vec{\mathbf{x}}', t_-) \frac{\partial}{\partial x^i}\left(\frac{1}{r}\right)\bigg|_t + \frac{1}{r}\frac{\partial}{\partial x^i}\rho(\vec{\mathbf{x}}', t_-)\bigg|_t \quad (4)$$

- Further using

$$\frac{\partial}{\partial x^i}\left(\frac{1}{r}\right)\bigg|_t = -\frac{r_i}{r^3},$$

$$\frac{\partial}{\partial x^i}\rho(\vec{\mathbf{x}}', t_-)\bigg|_t = \frac{\partial \rho}{\partial t_-}\frac{\partial t_-}{\partial r}\frac{\partial r}{dx^i}\bigg|_t = -\frac{\partial \rho(\vec{\mathbf{x}}', t_-)}{\partial t_-}\frac{1}{c}\frac{r_i}{r}, \quad (5)$$

- we get

$$\begin{aligned} \nabla\phi &= \frac{1}{4\pi\epsilon_0}\int\left[\rho(\vec{\mathbf{x}}', t_-)\hat{\mathbf{x}}^i \frac{\partial}{\partial x^i}\left(\frac{1}{r}\right)\bigg|_t + \frac{1}{r}\hat{\mathbf{x}}^i\frac{\partial}{\partial x^i}\rho(\vec{\mathbf{x}}, t_-)\bigg|_t\right] d^3x' \\ &= \frac{1}{4\pi\epsilon_0}\int\left[-\frac{\rho(\vec{\mathbf{x}}', t_-)}{r^2}\hat{\mathbf{r}} - \frac{\hat{\mathbf{r}}}{cr}\frac{\partial\rho(\vec{\mathbf{x}}', t_-)}{\partial t_-}\right] d^3x' \\ &= \frac{1}{4\pi\epsilon_0}\int\left[-\frac{[\rho(\vec{\mathbf{x}}')]}{r^2}\hat{\mathbf{r}} - \frac{[\dot{\rho}(\vec{\mathbf{x}}')]}{cr}\hat{\mathbf{r}}\right] d^3x' \end{aligned} \quad (6)$$

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$$= \frac{1}{4\pi\epsilon_0}\int\left[-\frac{\rho(\vec{\mathbf{x}}', t_-)}{r^2}\hat{\mathbf{r}} - \frac{\hat{\mathbf{r}}}{cr}\frac{\partial\rho(\vec{\mathbf{x}}', t_-)}{\partial t_-}\right] d^3x'$$

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Intermediate derivatives: curl of $\vec{\mathbf{A}}$

- Similar to the case of derivatives of ρ, we have

$$\frac{\partial}{\partial x^i}\vec{\mathbf{J}}(\vec{\mathbf{x}}', t_-)\Big|_t = \frac{\partial \vec{\mathbf{J}}}{\partial t_-}\frac{\partial t_-}{\partial r}\frac{\partial r}{dx^i}\Big|_t = -\frac{\partial \vec{\mathbf{J}}(\vec{\mathbf{x}}', t_-)}{\partial t_-}\frac{1}{c}\frac{r_i}{r}, \quad (7)$$

- And then

$$\begin{aligned}\nabla\times\vec{\mathbf{A}} &= \frac{\mu_0}{4\pi}\int\left[\hat{\mathbf{x}}^i\frac{\partial}{\partial x^i}\left(\frac{1}{r}\right)\times\vec{\mathbf{J}}(\vec{\mathbf{x}}', t_-) + \frac{1}{r}\hat{\mathbf{x}}^i\frac{\partial}{dx^i}\times\vec{\mathbf{J}}(\vec{\mathbf{x}}, t_-)\right]_t d^3x' \\ &= \frac{\mu_0}{4\pi}\int\left[\frac{\vec{\mathbf{J}}(\vec{\mathbf{x}}', t_-)}{r^2}\times\hat{\mathbf{r}} + \frac{1}{cr}\frac{\partial\vec{\mathbf{J}}(\vec{\mathbf{x}}', t_-)}{\partial t_-}\times\hat{\mathbf{r}}\right] d^3x' \\ &= \frac{\mu_0}{4\pi}\int\left[\frac{[\vec{\mathbf{J}}(\vec{\mathbf{x}}')]\times\hat{\mathbf{r}}}{r^2} + \frac{[\dot{\vec{\mathbf{J}}}(\vec{\mathbf{x}}')]\times\hat{\mathbf{r}}}{cr}\right] d^3x' \quad (8)\end{aligned}$$

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Electric and magnetic fields

- The electric field can now be calculated to be

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = -\nabla\phi - \frac{\partial \vec{\mathbf{A}}}{\partial t}$$

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0}\int\left(\frac{[\rho(\vec{\mathbf{x}}')]}{r^2}\hat{\mathbf{r}} + \frac{[\dot{\rho}(\vec{\mathbf{x}}')]}{cr}\hat{\mathbf{r}} - \frac{[\dot{\vec{\mathbf{J}}}(\mathbf{x}')]}{c^2 r}\right) d^3x' \quad (9)$$

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$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$: behaviour at large $|\vec{\mathbf{x}}|$

- In both, $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$, there are terms that behave as $1/r^2$ and there are terms that behave as $1/r$. The former are proportional to the sources, the latter are proportional to the rate of change of sources.
- When the sources are confined to a small region $|\vec{\mathbf{x}}'| < d$, then for $|\vec{\mathbf{x}}| >> d$, the $1/r$ terms dominate over the others. These are the "radiative" components of the fields.
- The radiative $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are then

$$\vec{\mathbf{E}}^{\rm rad}(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0} \int \left(\frac{[\dot{\rho}(\vec{\mathbf{x}}')]}{cr} \hat{\mathbf{r}} - \frac{[\dot{\vec{\mathbf{J}}}(\vec{\mathbf{x}}')]}{c^2 r} \right) d^3x' \qquad (11)$$

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- For sufficiently large r, we can take $\vec{\mathbf{r}} \approx \vec{\mathbf{x}}$.

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Simplification with monochromatic sources

- Monochromatic sources correspond to ("Real" part always implicit)

$$\rho(\vec{\mathbf{x}}', t) = \rho_0(\vec{\mathbf{x}}')e^{-i\omega t} , \quad (13)$$

$$\vec{\mathbf{J}}(\vec{\mathbf{x}}', t) = \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')e^{-i\omega t} . \quad (14)$$

- This dependence implies

$$\left[\rho(\vec{\mathbf{x}}')\right] = \rho(\vec{\mathbf{x}}', t_-) = \rho_0(\vec{\mathbf{x}}')e^{i(kr-\omega t)} , \quad (15)$$

$$\left[\vec{\mathbf{J}}(\vec{\mathbf{x}}')\right] = \vec{\mathbf{J}}(\vec{\mathbf{x}}', t_-) = \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')e^{i(kr-\omega t)} . \quad (16)$$

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- The potentials also then have the same frequency ω.

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$$[\vec{\mathbf{J}}(\vec{\mathbf{x}}')] = \vec{\mathbf{J}}(\vec{\mathbf{x}}', t_-) = \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')e^{i(kr-\omega t)} . \quad (16)$$

- And hence

$$\phi(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0}\int \rho_0(\vec{\mathbf{x}}')\frac{e^{i(kr-\omega t)}}{r}d^3x' ,$$
$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi}\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')\frac{e^{i(kr-\omega t)}}{r}d^3x' \quad (17)$$

- The potentials also then have the same frequency ω.

$\vec{\mathbf{E}}^{\rm rad}$ and $\vec{\mathbf{B}}^{\rm rad}$ in terms of $\dot{\vec{\mathbf{J}}}$ only

- Given the continuity equation

$$\nabla' \cdot \vec{\mathbf{J}}(\vec{\mathbf{x}}', t) + \dot{\rho}(\vec{\mathbf{x}}', t) = 0 \,, \tag{18}$$

it is clear that if $[\dot{\vec{\mathbf{J}}}(\vec{\mathbf{x}}')]$ is known everywhere, so is $[\dot{\rho}(\vec{\mathbf{x}}')]$, and the radiative $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields can be written in terms of $\vec{\mathbf{J}}$ only.

- For monochromatic sources, Some algebraic manipulation using the above result yields (See Panofsky-Phillips / Homework)

$$\vec{\mathbf{B}}^{\rm rad}(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0 c^3} \int \frac{[\dot{\vec{\mathbf{J}}}(\vec{\mathbf{x}}')] \times \hat{\mathbf{r}}}{r} d^3x' \tag{19}$$

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Note that $\mu_0/(4\pi) = 1/(4\pi\epsilon_0 c^2)$, one can use any combination.

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Frequency components of radiation fields

- Fourier components of the radiation fields are thus ($\vec{\mathbf{k}} = k\hat{\mathbf{r}}$) :

$$B_\omega^{rad}(\vec{\mathbf{x}}) = \frac{-i}{4\pi\epsilon_0 c^2} \int \left(\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}\right) \frac{e^{ikr}}{r} d^3x' \qquad (21)$$

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Outline

Energy from a radiation pulse

- The Poynting vector is $\vec{\mathbf{N}} = \vec{\mathbf{E}} \times \vec{\mathbf{H}}$, which gives the power radiated per unit area along it.
- Total energy radiated per unit area normal to $\vec{\mathbf{N}}$ is

$$\int_{-\infty}^{\infty} \vec{\mathbf{N}}(\vec{\mathbf{x}}, t)dt = \int_{-\infty}^{\infty} \vec{\mathbf{E}}(\vec{\mathbf{x}}, t) \times \vec{\mathbf{H}}(\vec{\mathbf{x}}, t)dt \quad (23)$$

$$= \int_{\omega,\omega',t=-\infty}^{\infty} \vec{\mathbf{E}}_{\omega}^{\mathrm{rad}} e^{-i\omega t} d\omega \times \vec{\mathbf{H}}_{\omega'}^{\mathrm{rad}} e^{-i\omega' t} d\omega' dt$$

$$= 2\pi \int_{0}^{\infty} \vec{\mathbf{E}}_{\omega}^{\mathrm{rad}} \times \vec{\mathbf{H}}_{-\omega}^{\mathrm{rad}} d\omega + 2\pi \int_{0}^{\infty} \vec{\mathbf{E}}_{-\omega}^{\mathrm{rad}} \times \vec{\mathbf{H}}_{\omega}^{\mathrm{rad}} d\omega \quad (24)$$

- But since $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{H}}(\vec{\mathbf{x}}, t)$ are real, $\vec{\mathbf{E}}_{-\omega} = \vec{\mathbf{E}}_{\omega}^{*}$ and $\vec{\mathbf{H}}_{-\omega} = \vec{\mathbf{H}}_{\omega}^{*}$.
- Then

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&= \int_{\omega,\omega',t=-\infty}^{\infty} \vec{\mathbf{E}}_{\omega}^{\mathrm{rad}} e^{-i\omega t} d\omega \times \vec{\mathbf{H}}_{\omega'}^{\mathrm{rad}} e^{-i\omega' t} d\omega' dt \\
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Calculating radiated energy for a pulse

- Substituting the expressions for $\vec{\mathbf{E}}^{\text{rad}}_\omega$ and $\vec{\mathbf{H}}^{\text{rad}}_\omega = \vec{\mathbf{B}}^{\text{rad}}_\omega/\mu_0$ obtained earlier, after a bit of algebra, gives

$$\vec{\mathbf{E}}^{\text{rad}}_\omega \times (\vec{\mathbf{H}}^{\text{rad}}_\omega)^* = \frac{1}{(4\pi)^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\left|\int \left(\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}\right)\frac{e^{ikr}}{r}d^3x'\right|^2 \hat{\mathbf{r}} \quad (26)$$

- Total radiated energy across a surface $d\vec{\mathbf{S}} = r^2\,\partial\Omega\,\hat{\mathbf{r}}$ is

$$U = 2\pi\int \vec{\mathbf{E}}^{\text{rad}*}_\omega \times (\vec{\mathbf{H}}^{\text{rad}}_\omega)^* d\omega \cdot r^2\, d\Omega\,\hat{\mathbf{r}} + c.c. \quad (27)$$

$$U = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\int_\omega \left|\int \left(\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}\right)\frac{e^{ikr}}{r}d^3x'\right|^2 d\omega r^2 d\Omega \quad (28)$$

- In other words,

$$\frac{dU_\omega}{d\Omega} = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\left|\int \left(\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}\right)e^{ikr}d^3x'\right|^2 \quad (29)$$

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Average power radiated by a monochromatic source

- Here we calculate the average radiated power /area over a cycle:

$$\frac{1}{T}\int_0^T \vec{\mathbf{N}}(\vec{\mathbf{x}}, t)dt = \frac{1}{T}\int_0^T \vec{\mathbf{E}}(\vec{\mathbf{x}}, t) \times \vec{\mathbf{H}}(\vec{\mathbf{x}}, t)dt \tag{30}$$

  where T is the periodicity of the wave.

- Since $\vec{\mathbf{E}}^{\mathrm{rad}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{E}}_0^{\mathrm{rad}} e^{-i\omega t}$ and $\vec{\mathbf{H}}^{\mathrm{rad}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{H}}_0^{\mathrm{rad}} e^{-i\omega t}$, the averaging gives

$$\begin{aligned}\langle N \rangle &= \frac{1}{2} E_0^{\mathrm{rad}} \times H_0^{\mathrm{rad}} \\ &= \frac{1}{2}\frac{1}{(4\pi)^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\left|\int \left(\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}\right)\frac{e^{ikr}}{r} d^3x'\right|^2 \hat{\mathbf{r}} \end{aligned} \tag{31}$$

- The average power radiated is then

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Long-distance approximation

- When $|\vec{\mathbf{x}}| >> |\vec{\mathbf{x}}'|$, then we have

$$kr = k|\vec{\mathbf{x}} - \vec{\mathbf{x}}'| \approx k|\vec{\mathbf{x}}| - \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}' \tag{33}$$

- Then for a radiation pulse,

$$\frac{dU_\omega}{d\Omega} = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\left|\int \left(\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}\right) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'} d^3x'\right|^2 \tag{34}$$

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This can be used in many instances, for example for radiation from antennas, as will be seen in later chapters.

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Recap of topics covered in this lecture

- $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields in the presence of moving sources
- Radiative components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$: the $1/r$ behaviour that dominates at large distances
- Poynting vector and power radiated by EM waves
- Long distance approximation for radiated power