Module I: Electromagnetic waves

Lecture 7: Multipole radiation, antennas

Amol Dighe
TIFR, Mumbai

Aug 26, 2018

Outline

Coming up...

Vector potential $\vec{\mathbf{A}}_0$ for monochromatic sources

- We are interested in calculating the radiative components of EM fields and related quantities (like radiated power) for a charge / current distribution that is oscillating with a frequency ω. The results for a general time dependence can be obtained by integrating over all frequencies (inverse Fourier transform).
- We have already seen that it is enough to know about the current distribution (we are interested only in radiative parts), since the charge distribution is related to it by continuity. In such a case, with $\vec{\mathbf{J}}(\vec{\mathbf{x}}', t) = \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')e^{-i\omega t}$, we get

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \qquad (1)$$

- Given $\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{A}}_0(\vec{\mathbf{x}})e^{-i\omega t}$, the rest of the quantities can be easily calculated in terms of it. We shall omit the “rad” label in this lecture, it is assumed to be everywhere except when specified.

Vector potential $\vec{\mathbf{A}}_0$ for monochromatic sources

- We are interested in calculating the radiative components of EM fields and related quantities (like radiated power) for a charge / current distribution that is oscillating with a frequency ω. The results for a general time dependence can be obtained by integrating over all frequencies (inverse Fourier transform).
- We have already seen that it is enough to know about the current distribution (we are interested only in radiative parts), since the charge distribution is related to it by continuity. In such a case, with $\vec{\mathbf{J}}(\vec{\mathbf{x}}', t) = \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')e^{-i\omega t}$, we get

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \qquad (1)$$

- Given $\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{A}}_0(\vec{\mathbf{x}})e^{-i\omega t}$, the rest of the quantities can be easily calculated in terms of it. We shall omit the “rad” label in this lecture, it is assumed to be everywhere except when specified.

Vector potential $\vec{\mathbf{A}}_0$ for monochromatic sources

- We are interested in calculating the radiative components of EM fields and related quantities (like radiated power) for a charge / current distribution that is oscillating with a frequency ω. The results for a general time dependence can be obtained by integrating over all frequencies (inverse Fourier transform).
- We have already seen that it is enough to know about the current distribution (we are interested only in radiative parts), since the charge distribution is related to it by continuity. In such a case, with $\vec{\mathbf{J}}(\vec{\mathbf{x}}', t) = \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')e^{-i\omega t}$, we get

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{1}$$

- Given $\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{A}}_0(\vec{\mathbf{x}})e^{-i\omega t}$, the rest of the quantities can be easily calculated in terms of it. We shall omit the “rad” label in this lecture, it is assumed to be everywhere except when specified.

$\vec{\mathbf{B}}^{\rm rad}$ and $\vec{\mathbf{E}}^{\rm rad}$ for monochromatic sources

- At large distances,

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} e^{i(k|x| - \omega t)} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \frac{e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \quad (2)$$

- Taking $\boxed{\vec{\mathbf{x}} = r\,\hat{\mathbf{r}}}$ and neglecting terms that go as $(1/r^2)$, the radiative part of the magnetic field is

$$\vec{\mathbf{B}}(\vec{\mathbf{x}}, t) = \nabla \times \vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = ik\hat{\mathbf{r}} \times \vec{\mathbf{A}}_0(\vec{\mathbf{x}}) e^{-i\omega t} \quad (3)$$

- The radiative part of the electric field can be obtained in this monochromatic case by using $(1/c^2)\partial\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)/\partial t = \nabla \times \vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$ (there is no current at large $|\vec{\mathbf{x}}|$):

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = \frac{c^2}{(-i\omega)} \nabla \times \vec{\mathbf{B}}(\vec{\mathbf{x}}, t) = c\vec{\mathbf{B}}_0(\vec{\mathbf{x}}) \times \hat{\mathbf{r}} e^{-i\omega t} \quad (4)$$

- Thus, $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$ fields are orthogonal to $\hat{\mathbf{r}}$, orthogonal to each other, and their magnitudes differ simply by a factor of c.

$\vec{\mathbf{B}}^{\rm rad}$ and $\vec{\mathbf{E}}^{\rm rad}$ for monochromatic sources

- At large distances,

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} e^{i(k|x| - \omega t)} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \frac{e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{2}$$

- Taking $\boxed{\vec{\mathbf{x}} = r\,\hat{\mathbf{r}}}$ and neglecting terms that go as $(1/r^2)$, the radiative part of the magnetic field is

$$\vec{\mathbf{B}}(\vec{\mathbf{x}}, t) = \nabla \times \vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = ik\hat{\mathbf{r}} \times \vec{\mathbf{A}}_0(\vec{\mathbf{x}}) e^{-i\omega t} \tag{3}$$

- The radiative part of the electric field can be obtained in this monochromatic case by using $(1/c^2)\partial\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)/\partial t = \nabla \times \vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$ (there is no current at large $|\vec{\mathbf{x}}|$):

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = \frac{c^2}{(-i\omega)} \nabla \times \vec{\mathbf{B}}(\vec{\mathbf{x}}, t) = c\vec{\mathbf{B}}_0(\vec{\mathbf{x}}) \times \hat{\mathbf{r}} e^{-i\omega t} \tag{4}$$

- Thus, $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$ fields are orthogonal to $\hat{\mathbf{r}}$, orthogonal to each other, and their magnitudes differ simply by a factor of c.

$\vec{B}^{\rm rad}$ and $\vec{E}^{\rm rad}$ for monochromatic sources

- At large distances,

$$\vec{\mathbf{A}}(\vec{\mathbf{x}},t) = \frac{\mu_0}{4\pi} e^{i(k|x|-\omega t)} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \frac{e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} d^3x' \qquad (2)$$

- Taking $\boxed{\vec{\mathbf{x}} = r\,\hat{\mathbf{r}}}$ and neglecting terms that go as $(1/r^2)$, the radiative part of the magnetic field is

$$\vec{\mathbf{B}}(\vec{\mathbf{x}},t) = \nabla\times\vec{\mathbf{A}}(\vec{\mathbf{x}},t) = ik\hat{\mathbf{r}}\times\vec{\mathbf{A}}_0(\vec{\mathbf{x}})e^{-i\omega t} \qquad (3)$$

- The radiative part of the electric field can be obtained in this monochromatic case by using $(1/c^2)\partial\vec{\mathbf{E}}(\vec{\mathbf{x}},t)/\partial t = \nabla\times\vec{\mathbf{B}}(\vec{\mathbf{x}},t)$ (there is no current at large $|\vec{\mathbf{x}}|$):

$$\vec{\mathbf{E}}(\vec{\mathbf{x}},t) = \frac{c^2}{(-i\omega)}\nabla\times\vec{\mathbf{B}}(\vec{\mathbf{x}},t) = c\vec{\mathbf{B}}_0(\vec{\mathbf{x}})\times\hat{\mathbf{r}}e^{-i\omega t} \qquad (4)$$

- Thus, $\vec{\mathbf{E}}(\vec{\mathbf{x}},t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}},t)$ fields are orthogonal to $\hat{\mathbf{r}}$, orthogonal to each other, and their magnitudes differ simply by a factor of c.

$\vec{\mathbf{B}}^{\rm rad}$ and $\vec{\mathbf{E}}^{\rm rad}$ for monochromatic sources

- At large distances,

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{\mu_0}{4\pi} e^{i(k|x|-\omega t)} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \frac{e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \qquad (2)$$

- Taking $\boxed{\vec{\mathbf{x}} = r\,\hat{\mathbf{r}}}$ and neglecting terms that go as $(1/r^2)$, the radiative part of the magnetic field is

$$\vec{\mathbf{B}}(\vec{\mathbf{x}}, t) = \nabla \times \vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = ik\hat{\mathbf{r}} \times \vec{\mathbf{A}}_0(\vec{\mathbf{x}}) e^{-i\omega t} \qquad (3)$$

- The radiative part of the electric field can be obtained in this monochromatic case by using $(1/c^2)\partial\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)/\partial t = \nabla \times \vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$ (there is no current at large $|\vec{\mathbf{x}}|$):

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = \frac{c^2}{(-i\omega)} \nabla \times \vec{\mathbf{B}}(\vec{\mathbf{x}}, t) = c\vec{\mathbf{B}}_0(\vec{\mathbf{x}}) \times \hat{\mathbf{r}} e^{-i\omega t} \qquad (4)$$

- Thus, $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$ fields are orthogonal to $\hat{\mathbf{r}}$, orthogonal to each other, and their magnitudes differ simply by a factor of c.

Long-distance, long-wavelength approximation

- For wavelength large compared to the source size, $|\vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'| \ll 1$, and we can expand the $e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}$ term (note that $\vec{\mathbf{k}} = k\hat{\mathbf{r}}$):

$$\vec{\mathbf{A}}_0(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi} e^{ikr} \sum \frac{(-ik)^n}{n!} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \frac{(\hat{\mathbf{r}} \cdot \vec{\mathbf{x}}')^n}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \qquad (5)$$

- If we approximate $|\vec{\mathbf{x}} - \vec{\mathbf{x}}'| \approx r$, i.e. neglect the corrections proportional to (d/r) where d is the source size, we get a simpler form

$$\vec{\mathbf{A}}_0(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum \frac{(-ik)^n}{n!} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')(\hat{\mathbf{r}} \cdot \vec{\mathbf{x}}')^n d^3x' \qquad (6)$$

  This is the approximate form of the "multipole expansion", and works for a few lower-order multipoles. Note that this approximation is fine as long as the expansion parameter in $|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|$ is much smaller than the expansion parameter in $e^{ik|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}$, i.e. $d/r << kd$, or $r \gg \lambda$.

- The complete expression for multipole expansion, valid even for intermediate distances, is given on the next page.

Long-distance, long-wavelength approximation

- For wavelength large compared to the source size, $|\vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'| \ll 1$, and we can expand the $e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}$ term (note that $\vec{\mathbf{k}} = k\hat{\mathbf{r}}$):

$$\vec{\mathbf{A}}_0(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi} e^{ikr} \sum \frac{(-ik)^n}{n!} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \frac{(\hat{\mathbf{r}} \cdot \vec{\mathbf{x}}')^n}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{5}$$

- If we approximate $|\vec{\mathbf{x}} - \vec{\mathbf{x}}'| \approx r$, i.e. neglect the corrections proportional to (d/r) where d is the source size, we get a simpler form

$$\vec{\mathbf{A}}_0(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum \frac{(-ik)^n}{n!} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')(\hat{\mathbf{r}} \cdot \vec{\mathbf{x}}')^n d^3x' \tag{6}$$

  This is the approximate form of the "multipole expansion", and works for a few lower-order multipoles. Note that this approximation is fine as long as the expansion parameter in $|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|$ is much smaller than the expansion parameter in $e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}$, i.e. $d/r << kd$, or $r \gg \lambda$.

- The complete expression for multipole expansion, valid even for intermediate distances, is given on the next page.

Long-distance, long-wavelength approximation

- For wavelength large compared to the source size, $|\vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'| \ll 1$, and we can expand the $e^{-i\vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'}$ term (note that $\vec{\mathbf{k}} = k\hat{\mathbf{r}}$):

$$\vec{\mathbf{A}}_0(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi} e^{ikr} \sum \frac{(-ik)^n}{n!} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \frac{(\hat{\mathbf{r}} \cdot \vec{\mathbf{x}}')^n}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3x' \tag{5}$$

- If we approximate $|\vec{\mathbf{x}} - \vec{\mathbf{x}}'| \approx r$, i.e. neglect the corrections proportional to (d/r) where d is the source size, we get a simpler form

$$\vec{\mathbf{A}}_0(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum \frac{(-ik)^n}{n!} \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') (\hat{\mathbf{r}} \cdot \vec{\mathbf{x}}')^n d^3x' \tag{6}$$

  This is the approximate form of the "multipole expansion", and works for a few lower-order multipoles. Note that this approximation is fine as long as the expansion parameter in $|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|$ is much smaller than the expansion parameter in $e^{ik|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}$, i.e. $d/r << kd$, or $r \gg \lambda$.

- The complete expression for multipole expansion, valid even for intermediate distances, is given on the next page.

Radiation potential at intermediate distances

- An expansion for $e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}/|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|$ exists in terms of Legendre polynomials, spherical Bessel functions and Hankel functions, which we give here without proof:

$$\frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = ik\sum(2n+1)P_n(\cos\theta')j_n(k|\vec{\mathbf{x}}'|)h_n(kr) \quad (7)$$

- At $k|\vec{\mathbf{x}}'| << 1$, we have

$$j_n(k|\vec{\mathbf{x}}'|) = \frac{2^n n!}{(2n+1)!}(k|\vec{\mathbf{x}}'|)^n \quad (8)$$

- For $kr >> 1$, we have

$$h_n(kr) = (-i)^{n+1}\frac{e^{ikr}}{kr} \quad (9)$$

- Using these two, the long-distance approximation gives

$$\frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = (-ik)^n\frac{e^{ikr}}{r}\sum\frac{2^n n!}{(2n)!}|\vec{\mathbf{x}}'|^n P_n(\cos\theta') \quad (10)$$

which matches our expansion to the two leading orders (check).

Radiation potential at intermediate distances

- An expansion for $e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}/|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|$ exists in terms of Legendre polynomials, spherical Bessel functions and Hankel functions, which we give here without proof:

$$\frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = ik\sum(2n+1)P_n(\cos\theta')j_n(k|\vec{\mathbf{x}}'|)h_n(kr) \qquad (7)$$

- At $k|\vec{\mathbf{x}}'| << 1$, we have

$$j_n(k|\vec{\mathbf{x}}'|) = \frac{2^n n!}{(2n+1)!}(k|\vec{\mathbf{x}}'|)^n \qquad (8)$$

- For $kr >> 1$, we have

$$h_n(kr) = (-i)^{n+1}\frac{e^{ikr}}{kr} \qquad (9)$$

- Using these two, the long-distance approximation gives

$$\frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = (-ik)^n\frac{e^{ikr}}{r}\sum\frac{2^n n!}{(2n)!}|\vec{\mathbf{x}}'|^n P_n(\cos\theta') \qquad (10)$$

which matches our expansion to the two leading orders (check).

Radiation potential at intermediate distances

- An expansion for $e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}/|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|$ exists in terms of Legendre polynomials, spherical Bessel functions and Hankel functions, which we give here without proof:

$$\frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = ik\sum(2n+1)P_n(\cos\theta')j_n(k|\vec{\mathbf{x}}'|)h_n(kr) \qquad (7)$$

- At $k|\vec{\mathbf{x}}'| << 1$, we have

$$j_n(k|\vec{\mathbf{x}}'|) = \frac{2^n n!}{(2n+1)!}(k|\vec{\mathbf{x}}'|)^n \qquad (8)$$

- For $kr >> 1$, we have

$$h_n(kr) = (-i)^{n+1}\frac{e^{ikr}}{kr} \qquad (9)$$

- Using these two, the long-distance approximation gives

$$\frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = (-ik)^n\frac{e^{ikr}}{r}\sum\frac{2^n n!}{(2n)!}|\vec{\mathbf{x}}'|^n P_n(\cos\theta') \qquad (10)$$

which matches our expansion to the two leading orders (check).

Radiation potential at intermediate distances

- An expansion for $e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}/|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|$ exists in terms of Legendre polynomials, spherical Bessel functions and Hankel functions, which we give here without proof:

$$\frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = ik\sum(2n+1)P_n(\cos\theta')j_n(k|\vec{\mathbf{x}}'|)h_n(kr) \qquad (7)$$

- At $k|\vec{\mathbf{x}}'| << 1$, we have

$$j_n(k|\vec{\mathbf{x}}'|) = \frac{2^n n!}{(2n+1)!}(k|\vec{\mathbf{x}}'|)^n \qquad (8)$$

- For $kr >> 1$, we have

$$h_n(kr) = (-i)^{n+1}\frac{e^{ikr}}{kr} \qquad (9)$$

- Using these two, the long-distance approximation gives

$$\frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = (-ik)^n\frac{e^{ikr}}{r}\sum\frac{2^n n!}{(2n)!}|\vec{\mathbf{x}}'|^n P_n(\cos\theta') \qquad (10)$$

which matches our expansion to the two leading orders (check).

Coming up...

The $n = 0$ term in the multipole expansion

- The leading ($n = 0$) term in the multipole expansion is

$$\vec{\mathbf{A}}_0^{(0)} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' \tag{11}$$

- The integral may be written in a more familiar form through the steps

$$\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' = -\int \vec{\mathbf{x}}'[\nabla' \cdot \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')]d^3x' \tag{12}$$

and using the continuity equation $\nabla' \cdot \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') = i\omega\rho_0(\vec{\mathbf{x}}')$:

$$\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' = -i\omega\int \vec{\mathbf{x}}'\rho_0(\vec{\mathbf{x}}')d^3\vec{\mathbf{x}}' = -i\omega\vec{\mathbf{p}} \tag{13}$$

where $\vec{\mathbf{p}}$ is the electric dipole moment.

- The $n = 0$ term thus represents the electric dipole radiation:

$$A_0^{ED}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\vec{\mathbf{p}} \tag{14}$$

The $n = 0$ term in the multipole expansion

- The leading ($n = 0$) term in the multipole expansion is

$$\vec{\mathbf{A}}_0^{(0)} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' \qquad (11)$$

- The integral may be written in a more familiar form through the steps

$$\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' = -\int \vec{\mathbf{x}}'[\nabla' \cdot \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')]d^3x' \qquad (12)$$

and using the continuity equation $\nabla' \cdot \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') = i\omega\rho_0(\vec{\mathbf{x}}')$:

$$\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' = -i\omega\int \vec{\mathbf{x}}'\rho_0(\vec{\mathbf{x}}')d^3\vec{\mathbf{x}}' = -i\omega\vec{\mathbf{p}} \qquad (13)$$

where $\vec{\mathbf{p}}$ is the electric dipole moment.

- The $n = 0$ term thus represents the electric dipole radiation:

$$A_0^{ED}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\vec{\mathbf{p}} \qquad (14)$$

The $n = 0$ term in the multipole expansion

- The leading ($n = 0$) term in the multipole expansion is

$$\vec{\mathbf{A}}_0^{(0)} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' \qquad (11)$$

- The integral may be written in a more familiar form through the steps

$$\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' = -\int \vec{\mathbf{x}}'[\nabla' \cdot \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')]d^3x' \qquad (12)$$

  and using the continuity equation $\nabla' \cdot \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') = i\omega\rho_0(\vec{\mathbf{x}}')$:

$$\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' = -i\omega\int \vec{\mathbf{x}}'\rho_0(\vec{\mathbf{x}}')d^3\vec{\mathbf{x}}' = -i\omega\vec{\mathbf{p}} \qquad (13)$$

  where $\vec{\mathbf{p}}$ is the electric dipole moment.

- The $n = 0$ term thus represents the electric dipole radiation:

$$A_0^{ED}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\vec{\mathbf{p}} \qquad (14)$$

The $n = 0$ term in the multipole expansion

- The leading ($n = 0$) term in the multipole expansion is

$$\vec{\mathbf{A}}_0^{(0)} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' \quad (11)$$

- The integral may be written in a more familiar form through the steps

$$\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' = -\int \vec{\mathbf{x}}'[\nabla' \cdot \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')]d^3x' \quad (12)$$

and using the continuity equation $\nabla' \cdot \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') = i\omega\rho_0(\vec{\mathbf{x}}')$:

$$\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')d^3x' = -i\omega\int \vec{\mathbf{x}}'\rho_0(\vec{\mathbf{x}}')d^3\vec{\mathbf{x}}' = -i\omega\vec{\mathbf{p}} \quad (13)$$

where $\vec{\mathbf{p}}$ is the electric dipole moment.

- The $n = 0$ term thus represents the electric dipole radiation:

$$A_0^{ED}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\vec{\mathbf{p}} \quad (14)$$

Electric dipole: $\vec{\mathbf{B}}$, $\vec{\mathbf{E}}$ and radiated power

- The magnetic and electric fields can immediately be written as

$$\vec{\mathbf{B}}_0^{ED}(\vec{\mathbf{x}}) = ik\hat{\mathbf{r}} \times \vec{\mathbf{A}}_0^{ED}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(ck^2)\hat{\mathbf{r}} \times \vec{\mathbf{p}} \quad (15)$$

$$\vec{\mathbf{E}}_0^{ED}(\vec{\mathbf{x}}) = c\vec{\mathbf{B}}_0^{ED}(\vec{\mathbf{x}}) \times \hat{\mathbf{r}} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(c^2k^2)(\hat{\mathbf{r}} \times \vec{\mathbf{p}}) \times \hat{\mathbf{r}} \quad (16)$$

- The Poynting vector $\vec{\mathbf{N}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{E}}(\vec{\mathbf{x}}, t) \times \vec{\mathbf{H}}(\vec{\mathbf{x}}, t)$ is normal to both, $(\hat{\mathbf{r}} \times \vec{\mathbf{p}})$ and $[(\hat{\mathbf{r}} \times p) \times \hat{\mathbf{r}}]$, i.e. along $\hat{\mathbf{r}}$, as expected.

$$\langle \vec{\mathbf{N}}(\vec{\mathbf{x}}) \rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}k^4c^3|\hat{\mathbf{r}} \times \vec{\mathbf{p}}|^2\hat{\mathbf{r}} \quad (17)$$

$$= \frac{\mu_0}{32\pi^2 r^2}k^4c^3|\vec{\mathbf{p}}|^2 \sin^2\theta\, \hat{\mathbf{r}} \quad (18)$$

- The average power radiated per solid angle is then

$$\frac{dP}{d\Omega} = \langle \vec{\mathbf{N}} \rangle \cdot r^2\hat{\mathbf{r}} = \frac{\mu_0}{32\pi^2}k^4c^3|\vec{\mathbf{p}}|^2 \sin^2\theta \quad (19)$$

Electric dipole: $\vec{\mathbf{B}}$, $\vec{\mathbf{E}}$ and radiated power

- The magnetic and electric fields can immediately be written as

$$\vec{\mathbf{B}}_0^{ED}(\vec{\mathbf{x}}) = ik\hat{\mathbf{r}} \times \vec{\mathbf{A}}_0^{ED}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(ck^2)\hat{\mathbf{r}} \times \vec{\mathbf{p}} \quad (15)$$

$$\vec{\mathbf{E}}_0^{ED}(\vec{\mathbf{x}}) = c\vec{\mathbf{B}}_0^{ED}(\vec{\mathbf{x}}) \times \hat{\mathbf{r}} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(c^2k^2)(\hat{\mathbf{r}} \times \vec{\mathbf{p}}) \times \hat{\mathbf{r}} \quad (16)$$

- The Poynting vector $\vec{\mathbf{N}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{E}}(\vec{\mathbf{x}}, t) \times \vec{\mathbf{H}}(\vec{\mathbf{x}}, t)$ is normal to both, $(\hat{\mathbf{r}} \times \vec{\mathbf{p}})$ and $[(\hat{\mathbf{r}} \times p) \times \hat{\mathbf{r}}]$, i.e. along $\hat{\mathbf{r}}$, as expected.

$$\langle\vec{\mathbf{N}}(\vec{\mathbf{x}})\rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}k^4c^3|\hat{\mathbf{r}} \times \vec{\mathbf{p}}|^2\hat{\mathbf{r}} \quad (17)$$

$$= \frac{\mu_0}{32\pi^2 r^2}k^4c^3|\vec{\mathbf{p}}|^2 \sin^2\theta \; \hat{\mathbf{r}} \quad (18)$$

- The average power radiated per solid angle is then

$$\frac{dP}{d\Omega} = \langle\vec{\mathbf{N}}\rangle \cdot r^2\hat{\mathbf{r}} = \frac{\mu_0}{32\pi^2}k^4c^3|\vec{\mathbf{p}}|^2 \sin^2\theta \quad (19)$$

Electric dipole: $\vec{\mathbf{B}}$, $\vec{\mathbf{E}}$ and radiated power

- The magnetic and electric fields can immediately be written as

$$\vec{\mathbf{B}}_0^{ED}(\vec{\mathbf{x}}) = ik\hat{\mathbf{r}} \times \vec{\mathbf{A}}_0^{ED}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(ck^2)\hat{\mathbf{r}} \times \vec{\mathbf{p}} \quad (15)$$

$$\vec{\mathbf{E}}_0^{ED}(\vec{\mathbf{x}}) = c\vec{\mathbf{B}}_0^{ED}(\vec{\mathbf{x}}) \times \hat{\mathbf{r}} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(c^2k^2)(\hat{\mathbf{r}} \times \vec{\mathbf{p}}) \times \hat{\mathbf{r}} \quad (16)$$

- The Poynting vector $\vec{\mathbf{N}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{E}}(\vec{\mathbf{x}}, t) \times \vec{\mathbf{H}}(\vec{\mathbf{x}}, t)$ is normal to both, $(\hat{\mathbf{r}} \times \vec{\mathbf{p}})$ and $[(\hat{\mathbf{r}} \times p) \times \hat{\mathbf{r}}]$, i.e. along $\hat{\mathbf{r}}$, as expected.

$$\langle\vec{\mathbf{N}}(\vec{\mathbf{x}})\rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}k^4c^3|\hat{\mathbf{r}} \times \vec{\mathbf{p}}|^2\hat{\mathbf{r}} \quad (17)$$

$$= \frac{\mu_0}{32\pi^2 r^2}k^4c^3|\vec{\mathbf{p}}|^2\sin^2\theta\,\hat{\mathbf{r}} \quad (18)$$

- The average power radiated per solid angle is then

$$\frac{dP}{d\Omega} = \langle\vec{\mathbf{N}}\rangle \cdot r^2\hat{\mathbf{r}} = \frac{\mu_0}{32\pi^2}k^4c^3|\vec{\mathbf{p}}|^2\sin^2\theta \quad (19)$$

Electric dipole radiation: salient features

- The radiated power is proportional to the fourth power of frequency. This results in the blue colour of the sky: the sunlight induces dipoles in the air molecules, which then radiate, giving out more light at high frequencies, i.e. near the blue end of the spectrum.
- The angular dependance is $\sin^2\theta$, i.e. there is no radiation in the direction of the dipole, most of the radiation is in the equatorial plane.
- At large wavelengths ($\lambda \gg L$), antennas (discussed later) also emit dipole radiation.

Electric dipole radiation: salient features

- The radiated power is proportional to the fourth power of frequency. This results in the blue colour of the sky: the sunlight induces dipoles in the air molecules, which then radiate, giving out more light at high frequencies, i.e. near the blue end of the spectrum.
- The angular dependance is $\sin^2\theta$, i.e. there is no radiation in the direction of the dipole, most of the radiation is in the equatorial plane.
- At large wavelengths ($\lambda \gg L$), antennas (discussed later) also emit dipole radiation.

Electric dipole radiation: salient features

- The radiated power is proportional to the fourth power of frequency. This results in the blue colour of the sky: the sunlight induces dipoles in the air molecules, which then radiate, giving out more light at high frequencies, i.e. near the blue end of the spectrum.
- The angular dependance is $\sin^2\theta$, i.e. there is no radiation in the direction of the dipole, most of the radiation is in the equatorial plane.
- At large wavelengths ($\lambda \gg L$), antennas (discussed later) also emit dipole radiation.

Coming up...

$n = 1$ term in the multipole expansion

- The $n = 1$ term in the expansion is

$$A_0^{(1)}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-ik) \int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')(\hat{\mathbf{r}} \cdot \vec{\mathbf{x}}') d^3x' \qquad (20)$$

- We separate the components of the integral that are symmetric and antisymmetric in $\vec{\mathbf{x}}'$ and $\vec{\mathbf{J}}_0(\vec{\mathbf{x}}')$:

$$\begin{aligned} I_{MD} &= \int \frac{1}{2}[(\hat{\mathbf{r}} \cdot \vec{\mathbf{x}}')\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') - (\hat{\mathbf{r}} \cdot \vec{\mathbf{J}}_0(\vec{\mathbf{x}}'))\vec{\mathbf{x}}']d^3x' \\ &= \int \frac{1}{2}[\vec{\mathbf{x}}' \times \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')] \times \hat{\mathbf{r}}\, d^3x' \qquad (21) \\ I_{EQ} &= \int \frac{1}{2}[(\hat{\mathbf{r}} \cdot \vec{\mathbf{x}}')\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') + (\hat{\mathbf{r}} \cdot \vec{\mathbf{J}}_0(\vec{\mathbf{x}}'))\vec{\mathbf{x}}']d^3x' \qquad (22) \end{aligned}$$

- These two terms correspond to the magnetic dipole and the electric quadrupole components, respectively, as we shall see.

$n = 1$ term in the multipole expansion

- The $n = 1$ term in the expansion is

$$A_0^{(1)}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')d^3x' \qquad (20)$$

- We separate the components of the integral that are symmetric and antisymmetric in $\vec{\mathbf{x}}'$ and $\vec{\mathbf{J}}_0(\vec{\mathbf{x}}')$:

$$\begin{aligned} I_{MD} &= \int \frac{1}{2}[(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') - (\hat{\mathbf{r}}\cdot\vec{\mathbf{J}}_0(\vec{\mathbf{x}}'))\vec{\mathbf{x}}']d^3x' \\ &= \int \frac{1}{2}[\vec{\mathbf{x}}' \times \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')] \times \hat{\mathbf{r}}\, d^3x' \qquad (21) \\ I_{EQ} &= \int \frac{1}{2}[(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') + (\hat{\mathbf{r}}\cdot\vec{\mathbf{J}}_0(\vec{\mathbf{x}}'))\vec{\mathbf{x}}']d^3x' \qquad (22) \end{aligned}$$

- These two terms correspond to the magnetic dipole and the electric quadrupole components, respectively, as we shall see.

$n = 1$ term in the multipole expansion

- The $n = 1$ term in the expansion is

$$A_0^{(1)}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')d^3x' \tag{20}$$

- We separate the components of the integral that are symmetric and antisymmetric in $\vec{\mathbf{x}}'$ and $\vec{\mathbf{J}}_0(\vec{\mathbf{x}}')$:

$$\begin{aligned} I_{MD} &= \int \frac{1}{2}[(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') - (\hat{\mathbf{r}}\cdot\vec{\mathbf{J}}_0(\vec{\mathbf{x}}'))\vec{\mathbf{x}}']d^3x' \\ &= \int \frac{1}{2}[\vec{\mathbf{x}}' \times \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')] \times \hat{\mathbf{r}}\, d^3x' \qquad (21) \\ I_{EQ} &= \int \frac{1}{2}[(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') + (\hat{\mathbf{r}}\cdot\vec{\mathbf{J}}_0(\vec{\mathbf{x}}'))\vec{\mathbf{x}}']d^3x' \qquad (22) \end{aligned}$$

- These two terms correspond to the magnetic dipole and the electric quadrupole components, respectively, as we shall see.

Magnetic dipole radiation

- Since the magnetic dipole moment is defined as

$$\vec{\mathbf{m}} = \int \frac{1}{2}[\vec{\mathbf{x}}' \times \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')]d^3x' \tag{23}$$

the component of $\vec{\mathbf{A}}_0$ corresponding to I_{MD} becomes

$$\vec{\mathbf{A}}_0^{MD}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)(\vec{\mathbf{m}} \times \hat{\mathbf{r}}) \tag{24}$$

- This immediately leads to

$$\vec{\mathbf{B}}_0^{MD}(\vec{\mathbf{x}}) = (ik)\hat{\mathbf{r}} \times \vec{\mathbf{A}}_0^{MD}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}k^2\, \hat{\mathbf{r}} \times (\vec{\mathbf{m}} \times \hat{\mathbf{r}}) \tag{25}$$

$$\vec{\mathbf{E}}_0^{MD}(\vec{\mathbf{x}}) = c\vec{\mathbf{B}}_0^{MD}(\vec{\mathbf{x}}) \times \hat{\mathbf{r}} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}k^2\, [\hat{\mathbf{r}} \times (\vec{\mathbf{m}} \times \hat{\mathbf{r}})] \times \hat{\mathbf{r}} \tag{26}$$

- And the average power radiated per unit area is

$$\langle \vec{\mathbf{N}}(\vec{\mathbf{x}}) \rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}k^4 c^3 |\vec{\mathbf{m}}|^2 \sin^2\theta\, \hat{\mathbf{r}} \tag{27}$$

where θ is the angle between $\vec{\mathbf{m}}$ and $\hat{\mathbf{r}}$.

Magnetic dipole radiation

- Since the magnetic dipole moment is defined as

$$\vec{\mathbf{m}} = \int \frac{1}{2}[\vec{\mathbf{x}}' \times \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')]d^3x' \tag{23}$$

the component of $\vec{\mathbf{A}}_0$ corresponding to I_{MD} becomes

$$\vec{\mathbf{A}}_0^{MD}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)(\vec{\mathbf{m}} \times \hat{\mathbf{r}}) \tag{24}$$

- This immediately leads to

$$\vec{\mathbf{B}}_0^{MD}(\vec{\mathbf{x}}) = (ik)\hat{\mathbf{r}} \times \vec{\mathbf{A}}_0^{MD}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}k^2\,\hat{\mathbf{r}} \times (\vec{\mathbf{m}} \times \hat{\mathbf{r}}) \tag{25}$$

$$\vec{\mathbf{E}}_0^{MD}(\vec{\mathbf{x}}) = c\vec{\mathbf{B}}_0^{MD}(\vec{\mathbf{x}}) \times \hat{\mathbf{r}} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}k^2\,[\hat{\mathbf{r}} \times (\vec{\mathbf{m}} \times \hat{\mathbf{r}})] \times \hat{\mathbf{r}} \tag{26}$$

- And the average power radiated per unit area is

$$\langle\vec{\mathbf{N}}(\vec{\mathbf{x}})\rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}k^4c^3|\vec{\mathbf{m}}|^2\sin^2\theta\,\hat{\mathbf{r}} \tag{27}$$

where θ is the angle between $\vec{\mathbf{m}}$ and $\hat{\mathbf{r}}$.

Magnetic dipole radiation

- Since the magnetic dipole moment is defined as

$$\vec{\mathbf{m}} = \int \frac{1}{2}[\vec{\mathbf{x}}' \times \vec{\mathbf{J}}_0(\vec{\mathbf{x}}')]d^3x' \tag{23}$$

the component of $\vec{\mathbf{A}}_0$ corresponding to I_{MD} becomes

$$\vec{\mathbf{A}}_0^{MD}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)(\vec{\mathbf{m}} \times \hat{\mathbf{r}}) \tag{24}$$

- This immediately leads to

$$\vec{\mathbf{B}}_0^{MD}(\vec{\mathbf{x}}) = (ik)\hat{\mathbf{r}} \times \vec{\mathbf{A}}_0^{MD}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}k^2\,\hat{\mathbf{r}} \times (\vec{\mathbf{m}} \times \hat{\mathbf{r}}) \tag{25}$$

$$\vec{\mathbf{E}}_0^{MD}(\vec{\mathbf{x}}) = c\vec{\mathbf{B}}_0^{MD}(\vec{\mathbf{x}}) \times \hat{\mathbf{r}} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}k^2\,[\hat{\mathbf{r}} \times (\vec{\mathbf{m}} \times \hat{\mathbf{r}})] \times \hat{\mathbf{r}} \tag{26}$$

- And the average power radiated per unit area is

$$\langle\vec{\mathbf{N}}(\vec{\mathbf{x}})\rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}k^4c^3|\vec{\mathbf{m}}|^2\sin^2\theta\,\hat{\mathbf{r}} \tag{27}$$

where θ is the angle between $\vec{\mathbf{m}}$ and $\hat{\mathbf{r}}$.

Electric quadrupole radiation

- The remaining component of $A_0^{(1)}$ is the electric quadrupole part (as will be clear soon):

$$A_0^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int \frac{1}{2}[(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') + (\hat{\mathbf{r}}\cdot\vec{\mathbf{J}}_0(\vec{\mathbf{x}}'))\vec{\mathbf{x}}']d^3x'$$

$$A_\beta^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\frac{\hat{\mathbf{r}}_\alpha}{2}\int [x'_\beta\vec{\mathbf{J}}_\alpha + x'_\alpha\vec{\mathbf{J}}_\beta]d^3x' \tag{28}$$

- Use

$$x'_\alpha J_\beta + x'_\beta J_\alpha = x'_\alpha(\nabla'_\gamma x'_\beta)J_\gamma + x'_\beta(\nabla'_\gamma x'_\alpha)J_\gamma \tag{29}$$

$$= \nabla'_\gamma(x'_\alpha x'_\beta)J_\gamma - x'_\alpha x'_\beta(\nabla'_\gamma J_\gamma) \tag{30}$$

$$= \nabla'_\gamma(x'_\alpha x'_\beta)J_\gamma - i\omega\, x'_\alpha x'_\beta\, \rho_0 \tag{31}$$

- To get

$$A_\beta^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)\hat{\mathbf{r}}_\alpha}{2}\int x'_\alpha x'_\beta\, \rho_0(\vec{\mathbf{x}}')d^3x' \tag{32}$$

Electric quadrupole radiation

- The remaining component of $A_0^{(1)}$ is the electric quadrupole part (as will be clear soon):

$$A_0^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int \frac{1}{2}[(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') + (\hat{\mathbf{r}}\cdot\vec{\mathbf{J}}_0(\vec{\mathbf{x}}'))\vec{\mathbf{x}}']d^3x'$$

$$A_\beta^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\frac{\hat{\mathbf{r}}_\alpha}{2}\int [x'_\beta\vec{\mathbf{J}}_\alpha + x'_\alpha\vec{\mathbf{J}}_\beta]d^3x' \quad (28)$$

- Use

$$x'_\alpha J_\beta + x'_\beta J_\alpha = x'_\alpha(\nabla'_\gamma x'_\beta)J_\gamma + x'_\beta(\nabla'_\gamma x'_\alpha)J_\gamma \quad (29)$$

$$= \nabla'_\gamma(x'_\alpha x'_\beta)J_\gamma - x'_\alpha x'_\beta(\nabla'_\gamma J_\gamma) \quad (30)$$

$$= \nabla'_\gamma(x'_\alpha x'_\beta)J_\gamma - i\omega\, x'_\alpha x'_\beta\, \rho_0 \quad (31)$$

- To get

$$A_\beta^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)\hat{\mathbf{r}}_\alpha}{2}\int x'_\alpha x'_\beta\, \rho_0(\vec{\mathbf{x}}')d^3x' \quad (32)$$

Electric quadrupole radiation

- The remaining component of $A_0^{(1)}$ is the electric quadrupole part (as will be clear soon):

$$A_0^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int \frac{1}{2}[(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') + (\hat{\mathbf{r}}\cdot\vec{\mathbf{J}}_0(\vec{\mathbf{x}}'))\vec{\mathbf{x}}']d^3x'$$

$$A_\beta^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\frac{\hat{\mathbf{r}}_\alpha}{2}\int [x'_\beta\vec{\mathbf{J}}_\alpha + x'_\alpha\vec{\mathbf{J}}_\beta]d^3x' \quad (28)$$

- Use

$$x'_\alpha J_\beta + x'_\beta J_\alpha = x'_\alpha(\nabla'_\gamma x'_\beta)J_\gamma + x'_\beta(\nabla'_\gamma x'_\alpha)J_\gamma \quad (29)$$

$$= \nabla'_\gamma(x'_\alpha x'_\beta)J_\gamma - x'_\alpha x'_\beta(\nabla'_\gamma J_\gamma) \quad (30)$$

$$= \nabla'_\gamma(x'_\alpha x'_\beta)J_\gamma - i\omega\, x'_\alpha x'_\beta\, \rho_0 \quad (31)$$

- To get

$$A_\beta^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)\hat{\mathbf{r}}_\alpha}{2}\int x'_\alpha x'_\beta\, \rho_0(\vec{\mathbf{x}}')d^3x' \quad (32)$$

Electric dipole radiation

- Connecting to the standard form:

$$A_\beta^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)\hat{\mathbf{r}}_\alpha}{2}\int x'_\alpha x'_\beta\ \rho_0(\vec{\mathbf{x}}')d^3x' \quad (33)$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)}{2}\frac{1}{3}\left[\hat{\mathbf{r}}_\alpha Q_{\alpha\beta} + \hat{\mathbf{r}}_\beta\int r'^2\rho_0(\vec{\mathbf{x}}')d^3x'\right] \quad (34)$$

- Here,

$$Q_{\alpha\beta} \equiv \int (3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta})\rho_0(\vec{\mathbf{x}}')d^3x'\ , \quad (35)$$

the electric quadrupole moment.

- One often uses $\vec{\mathbf{Q}}(\hat{\mathbf{r}})$, the component of the electric quadrupole moment along $\hat{\mathbf{r}}$, i.e.

$$\vec{\mathbf{Q}}_\alpha(\hat{\mathbf{r}}) = \sum Q_{\alpha\beta}\hat{\mathbf{r}}_\beta\ , \quad (36)$$

to write

$$A_0^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)}{2}\frac{1}{3}\left[\vec{\mathbf{Q}}(\hat{\mathbf{r}}) + \hat{\mathbf{r}}\int r'^2\rho_0(\vec{\mathbf{x}}')d^3x'\right] \quad (37)$$

Electric dipole radiation

- Connecting to the standard form:

$$A_\beta^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)\hat{\mathbf{r}}_\alpha}{2}\int x'_\alpha x'_\beta\ \rho_0(\vec{\mathbf{x}}')d^3x' \quad (33)$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)}{2}\frac{1}{3}\left[\hat{\mathbf{r}}_\alpha Q_{\alpha\beta} + \hat{\mathbf{r}}_\beta\int r'^2\rho_0(\vec{\mathbf{x}}')d^3x'\right] \quad (34)$$

- Here,

$$Q_{\alpha\beta} \equiv \int (3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta})\rho_0(\vec{\mathbf{x}}')d^3x' \ , \quad (35)$$

the electric quadrupole moment.

- One often uses $\vec{\mathbf{Q}}(\hat{\mathbf{r}})$, the component of the electric quadrupole moment along $\hat{\mathbf{r}}$, i.e.

$$\vec{\mathbf{Q}}_\alpha(\hat{\mathbf{r}}) = \sum Q_{\alpha\beta}\hat{\mathbf{r}}_\beta \ , \quad (36)$$

to write

$$A_0^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)}{2}\frac{1}{3}\left[\vec{\mathbf{Q}}(\hat{\mathbf{r}}) + \hat{\mathbf{r}}\int r'^2\rho_0(\vec{\mathbf{x}}')d^3x'\right] \quad (37)$$

Electric dipole radiation

- Connecting to the standard form:

$$A_\beta^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)\hat{\mathbf{r}}_\alpha}{2}\int x'_\alpha x'_\beta\ \rho_0(\vec{\mathbf{x}}')d^3x' \quad (33)$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)}{2}\frac{1}{3}\left[\hat{\mathbf{r}}_\alpha Q_{\alpha\beta} + \hat{\mathbf{r}}_\beta\int r'^2\rho_0(\vec{\mathbf{x}}')d^3x'\right] \quad (34)$$

- Here,

$$Q_{\alpha\beta} \equiv \int(3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta})\rho_0(\vec{\mathbf{x}}')d^3x'\ , \quad (35)$$

the electric quadrupole moment.

- One often uses $\vec{\mathbf{Q}}(\hat{\mathbf{r}})$, the component of the electric quadrupole moment along $\hat{\mathbf{r}}$, i.e.

$$\vec{\mathbf{Q}}_\alpha(\hat{\mathbf{r}}) = \sum Q_{\alpha\beta}\hat{\mathbf{r}}_\beta\ , \quad (36)$$

to write

$$A_0^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)}{2}\frac{1}{3}\left[\vec{\mathbf{Q}}(\hat{\mathbf{r}}) + \hat{\mathbf{r}}\int r'^2\rho_0(\vec{\mathbf{x}}')d^3x'\right] \quad (37)$$

Electric quadrupole: $\vec{\mathbf{B}}$, $\vec{\mathbf{E}}$ and power radiated

- Now we can calculate $\vec{\mathbf{B}}_0(\vec{\mathbf{x}})$ and $\vec{\mathbf{E}}_0(\vec{\mathbf{x}})$:

$$B_0^{EQ}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{-ik^3c}{6}\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}}) \tag{38}$$

$$E_0^{EQ}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{-ik^3c^2}{6}(\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}})) \times \hat{\mathbf{r}} \tag{39}$$

- The average Poynting vector is

$$\langle \vec{\mathbf{N}}(\vec{\mathbf{x}}) \rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}\frac{k^6c^3}{36}|\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}})|^2\,\hat{\mathbf{r}} \tag{40}$$

- The average power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{4\pi}\frac{k^6c^3}{288}|\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}})|^2 \tag{41}$$

Electric quadrupole: $\vec{\mathbf{B}}$, $\vec{\mathbf{E}}$ and power radiated

- Now we can calculate $\vec{\mathbf{B}}_0(\vec{\mathbf{x}})$ and $\vec{\mathbf{E}}_0(\vec{\mathbf{x}})$:

$$B_0^{EQ}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{-ik^3c}{6}\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}}) \tag{38}$$

$$E_0^{EQ}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{-ik^3c^2}{6}(\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}})) \times \hat{\mathbf{r}} \tag{39}$$

- The average Poynting vector is

$$\langle \vec{\mathbf{N}}(\vec{\mathbf{x}}) \rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}\frac{k^6c^3}{36}|\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}})|^2\ \hat{\mathbf{r}} \tag{40}$$

- The average power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{4\pi}\frac{k^6c^3}{288}|\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}})|^2 \tag{41}$$

Electric quadrupole: $\vec{\mathbf{B}}$, $\vec{\mathbf{E}}$ and power radiated

- Now we can calculate $\vec{\mathbf{B}}_0(\vec{\mathbf{x}})$ and $\vec{\mathbf{E}}_0(\vec{\mathbf{x}})$:

$$B_0^{EQ}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{-ik^3c}{6}\hat{\mathbf{r}}\times\vec{\mathbf{Q}}(\hat{\mathbf{r}}) \tag{38}$$

$$E_0^{EQ}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{-ik^3c^2}{6}(\hat{\mathbf{r}}\times\vec{\mathbf{Q}}(\hat{\mathbf{r}}))\times\hat{\mathbf{r}} \tag{39}$$

- The average Poynting vector is

$$\langle\vec{\mathbf{N}}(\vec{\mathbf{x}})\rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}\frac{k^6c^3}{36}|\hat{\mathbf{r}}\times\vec{\mathbf{Q}}(\hat{\mathbf{r}})|^2\,\hat{\mathbf{r}} \tag{40}$$

- The average power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{4\pi}\frac{k^6c^3}{288}|\hat{\mathbf{r}}\times\vec{\mathbf{Q}}(\hat{\mathbf{r}})|^2 \tag{41}$$

Comment on Electric quadrupole radiation

- If the charge distribution is azimuthally symmetric, and has a reflection symmetry about z axis (spheroidal distribution is a special case of this), then

$$Q_{xy} = Q_{yz} = Q_{xz} = 0 \,, \quad Q_{xx} = Q_{yy} = Q_0 \quad Q_{zz} = -2Q_0 \qquad (42)$$

  In such a case, it can be shown that the power radiated is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{4\pi}\frac{k^6 c^3}{32}|Q_0|^2 \sin^2\theta \cos^2\theta \qquad (43)$$

  where θ is the angle between $\hat{\mathbf{r}}$ and $\vec{\mathbf{Q}}(\hat{\mathbf{r}})$.

- The gravitational radiation has a similar form to the electric quadrupole radiation, except one has to deal with time-dependent mass distribution rather than time-dependent charge distribution.

Comment on Electric quadrupole radiation

- If the charge distribution is azimuthally symmetric, and has a reflection symmetry about z axis (spheroidal distribution is a special case of this), then

$$Q_{xy} = Q_{yz} = Q_{xz} = 0\ , \quad Q_{xx} = Q_{yy} = Q_0 \quad Q_{zz} = -2Q_0 \tag{42}$$

In such a case, it can be shown that the power radiated is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{4\pi}\frac{k^6 c^3}{32}|Q_0|^2 \sin^2\theta \cos^2\theta \tag{43}$$

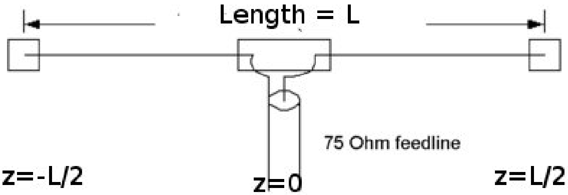
where θ is the angle between $\hat{\mathbf{r}}$ and $\vec{\mathbf{Q}}(\hat{\mathbf{r}})$.

- The gravitational radiation has a similar form to the electric quadrupole radiation, except one has to deal with time-dependent mass distribution rather than time-dependent charge distribution.

Coming up...

Dipole antenna from a coaxial cable

Feeding through a coaxial cable:



- An antenna may be represented as a monochromatic sinusoidal current with frequency ω:

$$\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') = \frac{I_0}{\sin(kL/2)}\delta(x')\delta(y')\sin\left[k\left(\frac{L}{2} - |z'|\right)\right]\hat{\mathbf{z}} \tag{44}$$

(Note: direction of current is the same for $z > 0$ and $z < 0$)

Calculating power radiated

- We have $\vec{\mathbf{x}}' = (0, 0, z')$. Using azimuthal symmetry, we choose $\hat{\mathbf{r}} = (\sin\theta, 0, \cos\theta)$, so that $\vec{\mathbf{k}} = k(\sin\theta, 0, \cos\theta)$, so that

$$\begin{aligned} \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \times \vec{\mathbf{k}} &= \frac{I_0 k \sin\theta}{\sin(kL/2)} \delta(x')\delta(y') \sin\left[k\left(\frac{L}{2} - |z'|\right)\right] \hat{\mathbf{y}} \\ \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}' &= z' \cos\theta \end{aligned} \quad (45)$$

- Then we get (Check on Mathematica)

$$\left| \int (\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'} d^3x' \right| = \frac{2I_0}{\sin\theta \sin(kL/2)} \left[\cos\left(\frac{kL\cos\theta}{2}\right) - \cos\left(\frac{kL}{2}\right)\right] \quad (46)$$

- This can be used to calculate the radiation pattern.

Calculating power radiated

- We have $\vec{\mathbf{x}}' = (0, 0, z')$. Using azimuthal symmetry, we choose $\hat{\mathbf{r}} = (\sin\theta, 0, \cos\theta)$, so that $\vec{\mathbf{k}} = k(\sin\theta, 0, \cos\theta)$, so that

$$\begin{aligned}\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \times \vec{\mathbf{k}} &= \frac{I_0 k \sin\theta}{\sin(kL/2)} \delta(x')\delta(y') \sin\left[k\left(\frac{L}{2} - |z'|\right)\right] \hat{\mathbf{y}} \\ \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}' &= z' \cos\theta \end{aligned} \quad (45)$$

- Then we get (Check on Mathematica)

$$\begin{aligned}&\left|\int (\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'} d^3x'\right| \\ &= \frac{2I_0}{\sin\theta \sin(kL/2)}\left[\cos\left(\frac{kL\cos\theta}{2}\right) - \cos\left(\frac{kL}{2}\right)\right] \end{aligned} \quad (46)$$

- This can be used to calculate the radiation pattern.

Calculating power radiated

- We have $\vec{\mathbf{x}}' = (0, 0, z')$. Using azimuthal symmetry, we choose $\hat{\mathbf{r}} = (\sin\theta, 0, \cos\theta)$, so that $\vec{\mathbf{k}} = k(\sin\theta, 0, \cos\theta)$, so that

$$\begin{aligned} \vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \times \vec{\mathbf{k}} &= \frac{I_0 k \sin\theta}{\sin(kL/2)} \delta(x')\delta(y') \sin\left[k\left(\frac{L}{2} - |z'|\right)\right] \hat{\mathbf{y}} \\ \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}' &= z' \cos\theta \end{aligned} \quad (45)$$

- Then we get (Check on Mathematica)

$$\begin{aligned} &\left| \int (\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'} d^3x' \right| \\ &= \frac{2I_0}{\sin\theta \sin(kL/2)} \left[\cos\left(\frac{kL\cos\theta}{2}\right) - \cos\left(\frac{kL}{2}\right) \right] \end{aligned} \quad (46)$$

- This can be used to calculate the radiation pattern.

Dependence of radiation pattern on kL

- In the large wavelength limit, $kL \ll 1 \Rightarrow$

$$\left| \int (\vec{\mathbf{J}}(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'} d^3x' \right| = \frac{I_0}{2} kL \sin\theta \qquad (47)$$

- The average power radiated is then

$$\begin{aligned} \frac{d\langle P \rangle}{d\Omega} &= \frac{1}{2}\frac{1}{(4\pi)^2}\sqrt{\frac{\mu_0}{\epsilon_0}} \left| \int \left(\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \times \vec{\mathbf{k}} \right) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'} d^3x' \right|^2 \\ &= \frac{1}{8}\frac{1}{(4\pi)^2}\sqrt{\frac{\mu_0}{\epsilon_0}} (I_0 kL)^2 \sin^2\theta \qquad (48) \end{aligned}$$

Dipole radiation pattern !

- As kL increases, stronger angular dependences appear. Check CDF demo.

Dependence of radiation pattern on kL

- In the large wavelength limit, $kL \ll 1 \Rightarrow$

$$\left| \int (\vec{\mathbf{J}}(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'} d^3x' \right| = \frac{I_0}{2} kL \sin\theta \tag{47}$$

- The average power radiated is then

$$\begin{aligned} \frac{d\langle P \rangle}{d\Omega} &= \frac{1}{2}\frac{1}{(4\pi)^2}\sqrt{\frac{\mu_0}{\epsilon_0}} \left| \int \left(\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \times \vec{\mathbf{k}} \right) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'} d^3x' \right|^2 \\ &= \frac{1}{8}\frac{1}{(4\pi)^2}\sqrt{\frac{\mu_0}{\epsilon_0}} (I_0 kL)^2 \sin^2\theta \end{aligned} \tag{48}$$

Dipole radiation pattern !

- As kL increases, stronger angular dependences appear. Check CDF demo.

Dependence of radiation pattern on kL

- In the large wavelength limit, $kL \ll 1 \Rightarrow$

$$\left| \int (\vec{\mathbf{J}}(\vec{\mathbf{x}}') \times \vec{\mathbf{k}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'} d^3x' \right| = \frac{I_0}{2} kL \sin\theta \tag{47}$$

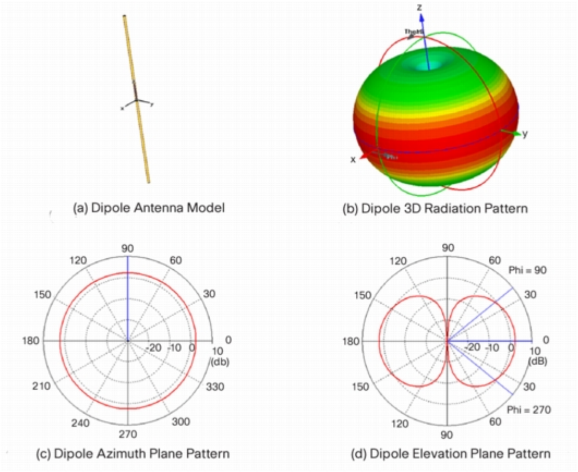
- The average power radiated is then

$$\begin{aligned} \frac{d\langle P\rangle}{d\Omega} &= \frac{1}{2}\frac{1}{(4\pi)^2}\sqrt{\frac{\mu_0}{\epsilon_0}} \left| \int \left(\vec{\mathbf{J}}_0(\vec{\mathbf{x}}') \times \vec{\mathbf{k}} \right) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'} d^3x' \right|^2 \\ &= \frac{1}{8}\frac{1}{(4\pi)^2}\sqrt{\frac{\mu_0}{\epsilon_0}} (I_0 kL)^2 \sin^2\theta \end{aligned} \tag{48}$$

Dipole radiation pattern !

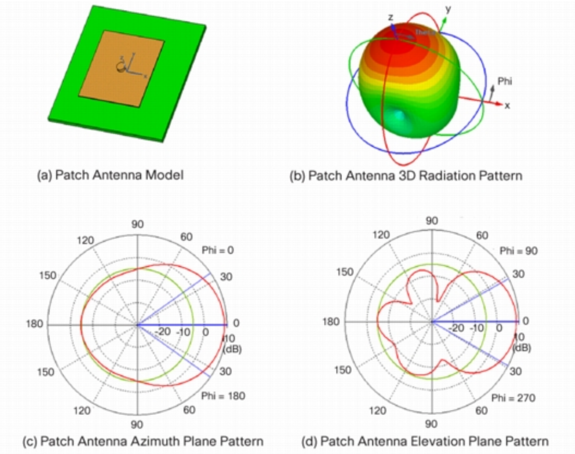
- As kL increases, stronger angular dependences appear. Check CDF demo.

Antenna patterns: dipole



(Note: units are dB)
http://www.cisco.com/en/US/prod/collateral/wireless/ps7183/ps469/prod_white_paper0900aecd806a1a3e.html

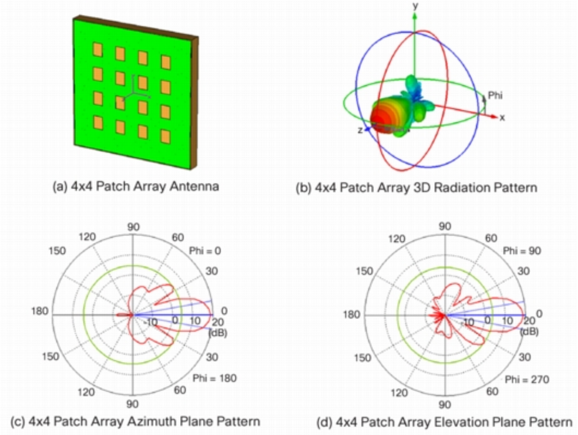
Antenna patterns: patch



(a) Patch Antenna Model
(b) Patch Antenna 3D Radiation Pattern
(c) Patch Antenna Azimuth Plane Pattern
(d) Patch Antenna Elevation Plane Pattern

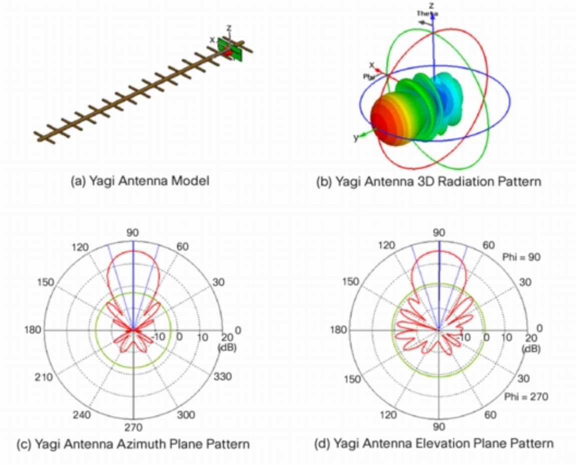
(Note: units are dB)
http://www.cisco.com/en/US/prod/collateral/wireless/ps7183/ps469/prod_white_paper0900aecd806a1a3e.html

Antenna patterns: patch array



(Note: units are dB)
http://www.cisco.com/en/US/prod/collateral/wireless/ps7183/ps469/prod_white_paper0900aecd806a1a3e.html

Antenna patterns: yagi



(a) Yagi Antenna Model

(b) Yagi Antenna 3D Radiation Pattern

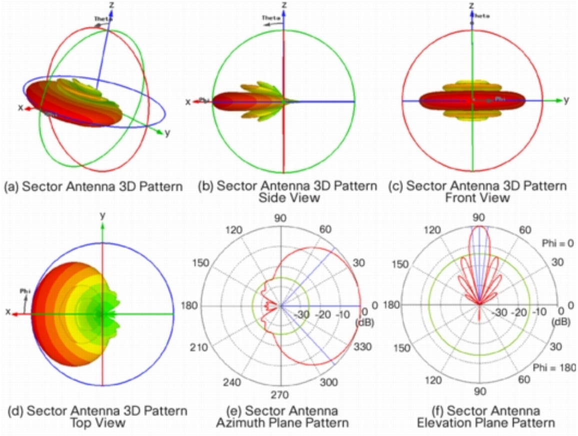
(c) Yagi Antenna Azimuth Plane Pattern

(d) Yagi Antenna Elevation Plane Pattern

(Note: units are dB)

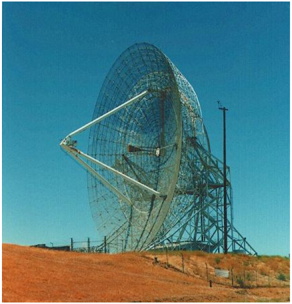
http://www.cisco.com/en/US/prod/collateral/wireless/ps7183/ps469/prod_white_paper0900aecd806a1a3e.html

Antenna patterns: sector

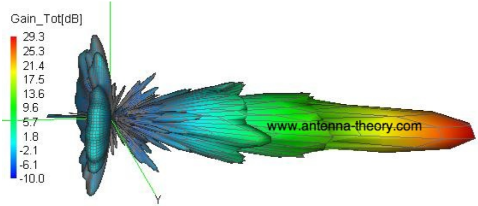


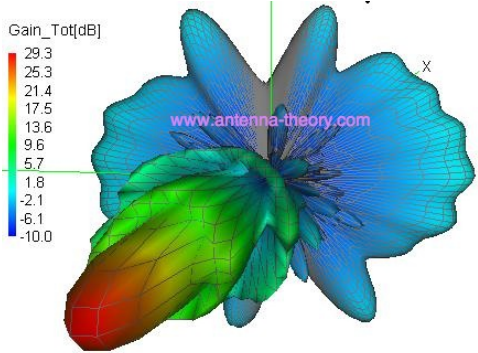
(Note: units are dB)
http://www.cisco.com/en/US/prod/collateral/wireless/ps7183/ps469/prod_white_paper0900aecd806a1a3e.html

Antenna patterns: dish



(Note: units are dB)





http://www.antenna-theory.com/antennas/main.php

Recap of topics covered in this lecture

- Calculating radiative components of $\vec{\mathbf{B}}$ and $\vec{\mathbf{E}}$ from $\vec{\mathbf{A}}$
- Multipole expansion when $|\vec{\mathbf{x}}'| < \lambda < |\vec{\mathbf{x}}|$
- Electric dipole radiation as the leading term $(n = 0)$ in multipole expansion
- Separating magnetic dipole moment and electric quadrupole moment contributions as antisymmetric and symmetric components of the subleading term $(n = 1)$
- $\vec{\mathbf{E}}$, $\vec{\mathbf{B}}$, Poynting vector, average rate of radiated power, and the angular distribution of radiated power for first few multipoles
- Antenna radiation patterns