Module II: Relativity and Electrodynamics

Lecture 8: From electrodynamics to Special Relativity

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Sep 3, 2018

Outline

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Motivations from Maxwell's equations

- The Lorentz force law seems to be valid whether the conductor is moving in a magnetic field, or a magnet is moving near a conductor, only the relative speeds count.
- With Galilean transformations $x' = x - vt$ and $t' = t$,

$$\left.\frac{\partial}{\partial x}\right|_t = \frac{\partial x'}{\partial x}\left.\frac{\partial}{\partial x'}\right|_{t'} + \frac{\partial t'}{\partial x}\left.\frac{\partial}{\partial t'}\right|_{x'} = \left.\frac{\partial}{\partial x'}\right|_{t'} \quad (1)$$

$$\left.\frac{\partial}{\partial t}\right|_x = \frac{\partial x'}{\partial t}\left.\frac{\partial}{\partial x'}\right|_{t'} + \frac{\partial t'}{\partial t}\left.\frac{\partial}{\partial t'}\right|_{x'} = \left.\frac{\partial}{\partial t'}\right|_{x'} - v\left.\frac{\partial}{\partial x'}\right|_{t'} \quad (2)$$

Maxwell's equations in vacuum:

$$\nabla \cdot \vec{\mathbf{E}} = 0 \qquad \nabla \times \vec{\mathbf{E}} = -\partial \vec{\mathbf{B}}/\partial t \quad (3)$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \qquad \nabla \times \vec{\mathbf{B}} = \mu_0 \epsilon_0\, \partial \vec{\mathbf{E}}/\partial t \quad (4)$$

- Do these remain the same in new Galilean coordinates ?

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Motivations from electromagnetic waves

- With Galilean transformations, does the wave equation for a scalar potential $\Phi(x, t)$,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi(x, t) = 0$$

remain invariant ? Is an EM wave in one frame is also an EM wave in another frame ? Moving with a different speed ?

- Indeed, the wave equation in vacuum for $\vec{\mathbf{E}}$ is

$$\nabla^2\vec{\mathbf{E}} + \mu_0\epsilon_0(\partial^2\vec{\mathbf{E}}/\partial t^2) = 0 \ ,$$

leads to a wave travelling with speed $c = 1/\sqrt{\mu_0\epsilon_0}$, which does not depend on the speed of the medium.

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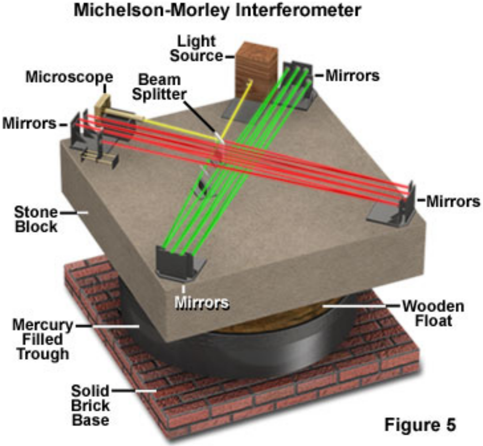
Motivations from measurements of the speed of light

Michaelson-Morley experiment (1887):



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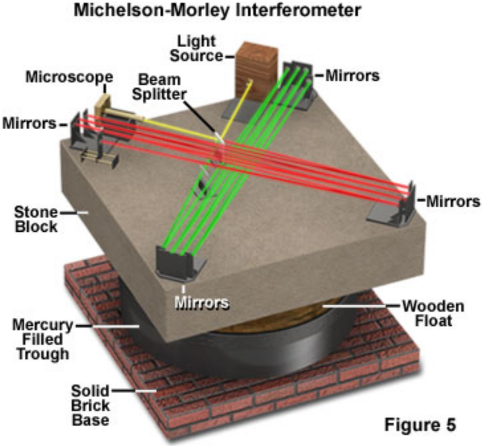

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An historical perspective

- After the Michaelson-Morley experiment, FitzGerald wrote a terse paper in [The Ether and the earth's atmosphere, Science 13: 390 (1889)], postulating that lengths may be contracted along the direction of movement through ether.
- In 1892, Lorentz wrote a more quantitative paper [In Dutch: De relatieve beweging van de aarde en den aether, Amsterdam, Zittingsverlag Akad. v. Wet., 1, p. 74 (1892)], in which he calculated that contraction by the factor $1 - v^2/(2c^2)$, would explain the MM experiment. This is the "Lorentz FitzGerald contraction".
- Einstein's celebrated paper ["Zur Elektrodynamik bewegter Körper", Annalen der Physik 322 (10), 1905], used the consistency of Maxwell's equations as his main motivation to come up with the revolutionary concept that space and time are unified. [Saha's translation: The Principle of Relativity: Original Papers by A. Einstein and H. Minkowski, University of Calcutta, 1920, pp. 1-34]

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Linear transformations of space and time

- A transformation of coordinates $(x, y, z, ct) \to (x', y', z', ct')$ should be linear, since it should not depend on where the origin of the coordinate system is. From another point of view, the infinitesimal transformation $(dx, dy, dz, c\,dt) \to (dx', dy', dz', c\,dt')$ has to be linear since the quadratic terms will vanish in the infinitesimal limit.
- For relative velocity $\vec{v}$ along x direction, one does not expect y and z to change, thus $y' = y$ and $z' = z$. For the other coordinates, one takes the most general transformation to be

$$x' = ax + bct\ , \quad ct' = px + qct\ . \tag{5}$$

- Let a beam of light starting at $(x, y, z) = (0, 0, 0)$ at $t = 0$ reach (x, y, z) at t. Then $(ct)^2 - x^2 - y^2 - z^2 = 0$. In the primed frame, this light has started at the origin at $t' = 0$, and has reached (x', y', z') at t'. Then $(ct')^2 - x'^2 - y'^2 - z'^2 = 0$. Combining these two gives

$$(p^2 - a^2 + 1)x^2 + (q^2 - b^2 - 1)c^2t^2 - 2(pq - ab)c\,t\,x = 0\ . \tag{6}$$

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Deriving Lorentz transformations

- Since the above expression has to be valid for alll x and t, all the three terms in Eq. 6 have to vanish individually. This allows us to substitute

$$p = \sinh\zeta_1 \ , \qquad a = \cosh\zeta_1 \ ,$$
$$q = \cosh\zeta_2 \ , \qquad b = \sinh\zeta_2 \ ,$$

to take care of the first two terms. The third term then yields $\sinh(\zeta_1 - \zeta_2) = 0$, i.e. $\zeta_1 = \zeta_2 = \zeta$. Thus,

$$p = b = \sinh\zeta \qquad q = a = \cosh\zeta \ .$$

- The coordinates then transform as

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Lorentz transformations in terms of β and γ

- Let S' be a frame moving with a speed v along x-axis, relative to an inertial frame S. The motion of the origin of S, as seen in S', is $x' = -vt'$.
- The transformations obtained earlier yield the coordinates of the origin of S, as seen in S', to be $x' = \sinh\zeta\, ct$, and $ct' = \cosh\zeta\, ct$.
- Thus, $\tanh\zeta = -v/c = -\beta$. This gives

$$\cosh\zeta = \frac{1}{\sqrt{1-\beta^2}} = \gamma\ , \quad \sinh\zeta = -\frac{\beta}{\sqrt{1-\beta^2}} = -\gamma\beta\ . \tag{8}$$

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Some remarks on Lorentz transformations

- The inverse Lorentz transformations are

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  Clearly, these are obtained simply by changing the sign of v or β.

- We have only talked about Lorentz transformations with "boosts" along x direction. However using the isotropy of space within each reference frame, we can always rotate the frames such that the relative motion between them is along the x axis. Formally speaking,

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R_1 & \\ 0 & & & \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R_2 & \\ 0 & & & \end{pmatrix} \qquad (11)$$

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Length contraction

- The question: If the length of an object measured in frame S is L, what is the length measured in S' ?
- Formal statement: An object is stationary in frame S. The coordinates of two ends of the object in this frame are x_1 and x_2, independent of t_1 or t_2. The measurement of length L corresponds to $x_2 - x_1 = L$. In frame S', the coordinates of these ends are (x_1', t_1') and (x_2', t_2'). The measurement of length in this frame corresponds to $L' = x_2' - x_1'$, when $t_1' = t_2'$.
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$$x_1 = \gamma x_1' + \gamma\beta c t_1' \ , \quad x_2 = \gamma x_2' + \gamma\beta c t_2' \ .$$

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Time dilation

- Formal statement: The time interval between two events at the same location in frame S (at $x_1 = x_2$) is $T = t_2 - t_1$, what is $t'_2 - t'_1$ in frame S' ?
- We have

$$ct'_1 = \gamma c t_1 - \gamma\beta x_1 \ , \quad ct'_2 = \gamma c t_2 - \gamma\beta x_2 \ .$$

- Since $x_1 = x_2$, this gives

$$T' = (t'_2 - t'_1) = \gamma(t_2 - t_1) = \gamma T \ , \qquad (13)$$

which is time dilation.

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- Formal statement: The time interval between two events at the same location in frame S (at $x_1 = x_2$) is $T = t_2 - t_1$, what is $t_2' - t_1'$ in frame S' ?
- We have

$$ct_1' = \gamma ct_1 - \gamma\beta x_1 \ , \quad ct_2' = \gamma ct_2 - \gamma\beta x_2 \ .$$

- Since $x_1 = x_2$, this gives

$$T' = (t_2' - t_1') = \gamma(t_2 - t_1) = \gamma T \ , \tag{13}$$

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Velocity measured from a moving frame

- Let the velocity of a particle in the frame S be (u_x, u_y, u_z). What will be the velocity of this particle as measured in a frame S' moving with a speed v in the x-direction (in the S frame) ?
- Formalizing: Given $dx/dt = u_x, dy/dt = u_y, dz/dt = u_z$, determine $u'_x = dx'/dt', u'_y = dy'/dt', u'_z = dz'/dt'$.
- Using the Lorentz transformations

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Velocity addition

- The results about velocities measured in different frames lead directly to the “velocity addition” formula: If two particles A and B are moving *towards* each other with speeds u_A and u_B, then the speed of approach of A as measured by B is obtained simply by using $v = -u_B$ in the earlier results.

- This gives

$$u_A \oplus u_B = \frac{u_A + u_B}{1 + u_A u_B / c^2} . \tag{15}$$

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Acceleration

- In frame S, a particle has instantaneous velocity (u_x, u_y, u_z) and acceleration (a_x, a_y, a_z). What is the velocity and acceleration in frame S' (moving with speed v along the x-direction) ?
- The velocity in the new frame is (u'_x, u'_y, u'_z) as calculated earlier.
- The acceleration components are obtained simply through

$$a'_x = \frac{du'_x}{dt'}\,, \quad a'_y = \frac{du'_y}{dt'}\,, \quad a'_z = \frac{du'_z}{dt'}\,,$$

where one has to use $du_x/dt = a_x, du_y/dt = a_y, du_z/dt = a_z$.

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Take-home message from this lecture

- Even without the actual measurement of the speed of light, the invariance of Maxwell's equations indicates that the speed of light should be independent of the motion of the medium.
- Given the speed of light is the same in all frames, there is a unique set of linear transformations that achieves this, which are Lorentz transformations.
- The transformations of velocity and acceleration while going from one inertial frame to another