Module II: Relativity and Electrodynamics

Lecture 9: Transformations of EM fields and waves; Relativistic momentum and energy

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Outline

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Invariance of Maxwell's equations

- We would like Maxwell's equations to be invariant under Lorentz transformations. That is, in vacuum, we would like to have

$$\begin{aligned} \nabla' \cdot \vec{\mathbf{E}}' &= 0 \,, & \nabla' \times \vec{\mathbf{E}}' &= -\partial \vec{\mathbf{B}}'/\partial t' \,, \\ \nabla' \cdot \vec{\mathbf{B}}' &= 0 \,, & \nabla' \times \vec{\mathbf{B}}' &= \mu_0 \epsilon_0 \partial \vec{\mathbf{E}}'/\partial t' \,. \end{aligned} \tag{1}$$

- This condition allows us to determine the transformation properties of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$. (See Einstein's original paper.)
- From the Lorentz transformations and the chain rule for derivatives,

$$\begin{aligned} \frac{\partial}{\partial(ct')} &= \frac{\partial x}{\partial(ct')}\frac{\partial}{\partial x} + \frac{\partial(ct)}{\partial(ct')}\frac{\partial}{\partial(ct)} = \gamma\beta\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial(ct)} \\ \frac{\partial}{\partial x'} &= \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial(ct)}{\partial x'}\frac{\partial}{\partial(ct)} = \gamma\frac{\partial}{\partial x} + \gamma\beta\frac{\partial}{\partial(ct)} \\ \frac{\partial}{\partial y'} &= \frac{\partial}{\partial y} \,, \quad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z} \end{aligned} \tag{2}$$

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Transformations of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields

Problem

The transformations for components of ∇' and $\partial/\partial t'$ were obtained earlier in this lecture, in eq. 2. Assume the components of $\vec{\mathbf{E}}'$ and $\vec{\mathbf{B}}'$ to be some linear combinations of the components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$, with coefficients that are functions of v. Show that

$$E'_x = E_x \ , \quad E'_y = \gamma(E_y - vB_z) \ , \quad E'_z = \gamma(E_z + vB_y) \ ,$$
$$B'_x = B_x \ , \quad B'_y = \gamma(B_y + \frac{v}{c^2} E_z) \ , \quad B'_z = \gamma(B_z - \frac{v}{c^2} E_y) \ .$$

Lorentz force from relativistic transformations

- We thus know how $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields should transform under a change of frame. The transformations can be written in the form:

$$\vec{\mathbf{E}}'_{\parallel} = \vec{\mathbf{E}}_{\parallel} \; , \qquad \vec{\mathbf{E}}'_{T} = \gamma(\vec{\mathbf{E}}_{T} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}_{T}) \; ,$$

$$\vec{\mathbf{B}}'_{\parallel} = \vec{\mathbf{B}}_{\parallel} \; , \qquad \vec{\mathbf{B}}'_{T} = \gamma(\vec{\mathbf{B}}_{T} - \frac{\vec{\mathbf{v}}}{c^2} \times \vec{\mathbf{E}}_{T}) \; ,$$

  where the subscript T denotes components transverse to the relative motion of frames, and $\parallel$ denotes components along the relative motion of frames.

- Note that in the limit of small velocities ($\gamma \approx 1$), the above set of equations give

$$\vec{\mathbf{E}}' = \vec{\mathbf{E}}'_{\parallel} + \vec{\mathbf{E}}'_{T} \approx \vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}} \; , \qquad (3)$$

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More about the Lorentz force

- Relativity tells us that Lorentz force law is not an arbitrary addition to Maxwell's equations, but just a consequence of the additional principle of invariance of Maxwell's equations under frame changes.
- It shows that the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are connected. A pure $\vec{\mathbf{B}}$ field in a frame give rise to an $\vec{\mathbf{E}}'$ field in another. On the other hand, magnetic fields $\vec{\mathbf{B}}'$ may be produced by purely electric fields $\vec{\mathbf{E}}$ in another inertial frame.
- The standard Lorentz force law is valid only at low velocities. At large velocities, the law "$\vec{\mathbf{E}}' = \vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}$" has to be modified to take care of the factors of γ in some components.

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- This gives

$$\frac{\partial^2}{\partial(ct)^2} - \nabla^2 = \frac{\partial^2}{\partial(ct')^2} - \nabla'^2\,, \qquad (5)$$

so that the electromagnetic fields in free space, which are solutions of

$$\left(\frac{\partial^2}{\partial(ct)^2} - \nabla^2\right)\vec{\mathbf{V}}(\vec{\mathbf{x}}, t) = 0$$

are the solutions of

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Invariance of the plane wave solution

- Since the wave equation does not change, the plane wave solution in one frame stays a plane wave solution in another frame. That is, if a plane wave has the form

$$\vec{\mathbf{V}}_0 \exp[i(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}} - \omega t)]\text{ , with } \omega = c|\vec{\mathbf{k}}|$$

in one frame S, it takes the form

$$\vec{\mathbf{V}}'_0 \exp[i(\vec{\mathbf{k}}'\cdot\vec{\mathbf{x}}' - \omega' t')]\text{ , with } \omega' = c|\vec{\mathbf{k}}'|$$

in the other frame S'.

- The relationship between the primed and unprimed values of k and ω can be obtained by equating the phases at (x, y, z, t) and the corresponding (x', y', z', t').
- If the frame S' is moxing with speed v along the x direction, then for a wave with $\vec{\mathbf{k}} = (k_x, k_y, k_z)$ and $\vec{\mathbf{k}}' = (k'_x, k'_y, k'_z)$, the phase-equality $\vec{\mathbf{k}}\cdot\vec{\mathbf{x}} - \omega t = \vec{\mathbf{k}}'\cdot\vec{\mathbf{x}}' - \omega' t'$ gives

$$\begin{aligned} k_x x + k_y y + k_z z - |\vec{\mathbf{k}}|ct &= \\ &(\gamma k'_x + \gamma\beta|\vec{\mathbf{k}}'|)x + k'_y y + k'_z z - (\gamma\beta k'_x + \gamma|\vec{\mathbf{k}}'|)ct \,. \end{aligned} \qquad (6)$$

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Aberration

- Let a source be stationary in the frame S', i.e. it is moving with a speed v in the x-direction in the S frame. Let the wavevector in the S' frame be in the $x' - y'$ plane, making an angle θ' with the x'-axis. That is, $\tan\theta' = k'_y/k'_x$.
- As seen from frame S, the wavevector makes an angle θ with the x-axis, with

$$\tan\theta = \frac{k_y}{k_x} = \frac{k'_y}{\gamma k'_x + \gamma\beta|\vec{\mathbf{k}}'|} = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)} \tag{7}$$

- Thus the direction of emission of an EM wave from a source changes if the source is moving. This is the phenomenon of aberration.
- Note that $\theta' = 0 \Rightarrow \theta = 0$, $\theta' = \pi \Rightarrow \theta = \pi$, however $\theta' = \pi/2 \Rightarrow \theta = 1/\gamma\beta$. Thus, aberration tends to focus the directions of emitted waves towards the direction of motion of the source.

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Doppler effect

- Equating the coefficients of (ct) on the two sides of the phase-equality, one gets

$$|\vec{\mathbf{k}}| = \gamma\beta k'_x + \gamma|\vec{\mathbf{k}}'| = \gamma(1+\beta\cos\theta')|\vec{\mathbf{k}}'| \qquad (8)$$

- Thus, $\omega = \omega'\gamma(1+\beta\cos\theta')$. This change of frequency due to the motion of the source is Doppler effect.
- Note the three special cases:

$$\begin{aligned} \theta' = 0 \quad &\Rightarrow \quad \omega = \omega'\sqrt{(1+\beta)/(1-\beta)}\,, \\ \theta' = \pi \quad &\Rightarrow \quad \omega = \omega'\sqrt{(1-\beta)/(1+\beta)}\,, \\ \theta' = \pi/2 \quad &\Rightarrow \quad \omega = \omega'\gamma\,. \end{aligned}$$

The first two cases are the blue-shift and red-shift associated with approaching and receding sources, respectively, present even without relativity (though the magnitudes are different). The third is the transverse Doppler shift, which is absent if relativistic effects are ignored.

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Intensity of the wave

- Intensity ⇒ power transmitted ⇒ Poynting vector
- The magnitude of the Poynting vector $\vec{\mathbf{N}}$ is

$$|\vec{\mathbf{N}}| = |\vec{\mathbf{E}} \times \vec{\mathbf{H}}| = \frac{1}{\mu_0 c}|E|^2 = \frac{1}{\mu_0 c}(|\vec{\mathbf{E}}_{||}|^2 + |\vec{\mathbf{E}}_T|^2) \tag{9}$$

- We know that

$$\vec{\mathbf{E}}_{||} = \vec{\mathbf{E}}'_{||}\ , \quad \vec{\mathbf{E}}_T = \gamma(\vec{\mathbf{E}}'_T - \vec{\mathbf{v}} \times \vec{\mathbf{B}}'_T) \tag{10}$$

- Let us define $\vec{\mathbf{k}}'$ to be in the x-y plane.
  (Directions of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ do not change by $\hat{\mathbf{x}}$-boost.)
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- Let us define $\vec{\mathbf{k}}'$ to be in the x-y plane.
  (Directions of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ do not change by $\hat{\mathbf{x}}$-boost.)
- separate the $\vec{\mathbf{E}}'$ into two components:
  - Component in the x-y plane
  - Component normal to the x-y plane

$\vec{\mathbf{k}}$ and $\vec{\mathbf{E}}$ in the x-y plane

- The components of $\vec{\mathbf{k}}'$, $\vec{\mathbf{E}}'$, $\vec{\mathbf{B}}'$ are

$$\begin{aligned} \vec{\mathbf{k}}' &= |\vec{\mathbf{k}}'| \cos\theta' \ \hat{\mathbf{x}} + |\vec{\mathbf{k}}'| \sin\theta' \ \hat{\mathbf{y}} \\ \vec{\mathbf{E}}' &= -|\vec{\mathbf{E}}'| \sin\theta' \ \hat{\mathbf{x}} + |\vec{\mathbf{E}}'| \cos\theta' \ \hat{\mathbf{y}} \\ \vec{\mathbf{B}}' &= (|\vec{\mathbf{E}}'|/c) \ \hat{\mathbf{z}} \end{aligned} \quad (11)$$

- The Poynting vector becomes

$$\begin{aligned} |\vec{\mathbf{N}}| &= \frac{1}{\mu_0 c} |\vec{\mathbf{E}}'|^2 \left[\sin^2\theta' + \gamma^2 \left(\cos\theta' + \frac{v}{c} \right)^2 \right] \\ &= |\vec{\mathbf{N}}'| \left[\sin^2\theta' + \gamma^2 \left(\cos\theta' + \frac{v}{c} \right)^2 \right] \end{aligned} \quad (12)$$

- Enhancement along the direction of motion of the source

Problem

Repeat the calculation for $\vec{\mathbf{E}}'$ along the z axis

$\vec{\mathbf{k}}$ and $\vec{\mathbf{E}}$ in the x-y plane

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Recap of transformations of EM fields

- Transformations of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields, and derivation of Lorentz force law from Maxwell's equations
- Relativistic aberration, Doppler shift, and change in intensity of an EM wave
- See Anthony Searle simulations

Outline

Desirable properties of relativistic momentum

- In the non-relativistic world, momentum is simply given by $\vec{\mathbf{p}} = m\vec{\mathbf{u}}$, where $\vec{\mathbf{u}}$ is the velocity of the object. This does not work with relativity (as will be seen in the following example), and we have to go for a generalization of this.
- We look for a generalized definition of momentum of the form

$$\vec{\mathbf{p}}(\vec{\mathbf{u}}) = \mathcal{M}(u)\vec{\mathbf{u}} \,, \quad \text{with} \quad \lim_{u \to 0} \mathcal{M}(u) = m \,, \tag{13}$$

  where m is the mass of the particle. Note that we have still kept certain desirable properties of momentum:
  (i) $\vec{\mathbf{p}}$ is in the same direction as $\vec{\mathbf{u}}$.
  (ii) The function $\mathcal{M}(u)$ depends only on the magnitude of $\vec{\mathbf{u}}$, and is independent of its direction. This corresponds to isotropy of space.

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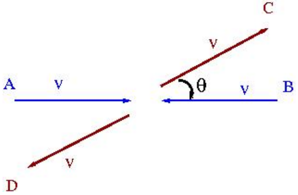
Elastic collision of same masses and speeds

- Consider the elastic collision of two objects A and B of the same mass m, moving towards each other with speed v. They undergo an elastic collision such that each of them gets deflected through an angle θ, without any change in the speeds. Let us call A and B after collision by C and D, respectively, for convenience.



- The momenta of A, B, C, D are then

$$\vec{\mathbf{p}}_A = \mathcal{M}(v)(v, 0, 0) \ , \qquad \vec{\mathbf{p}}_C = \mathcal{M}(v)(v\cos\theta, v\sin\theta, 0) \ ,$$
$$\vec{\mathbf{p}}_B = \mathcal{M}(v)(-v, 0, 0) \ , \qquad \vec{\mathbf{p}}_D = \mathcal{M}(v)(-v\cos\theta, -v\sin\theta, 0) \ .$$

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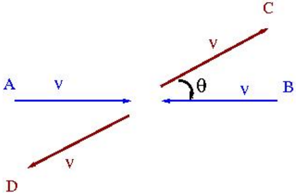

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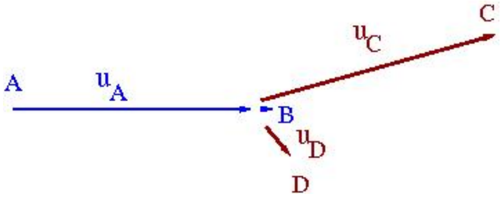
Elastic collision in the frame of a stationary mass

- Let us now see the above collision in the frame S' which is moving with a speed *v* along *x* direction. In this frame, B is initially stationary.



Momenta in the S' frame

- Since we know how velocities transform under frame change, and since $\vec{\mathbf{p}}$ is a function only of the velocities, we can write down the momenta of the particles in frame S' as

$$
\begin{aligned}
\vec{\mathbf{p}}'_A &= \mathcal{M}(u_A)\left(\frac{2v}{1+v^2/c^2}, 0, 0\right) \ , \\
\vec{\mathbf{p}}'_B &= (0,0,0) \ , \\
\vec{\mathbf{p}}'_C &= \mathcal{M}(u_C)\left(\frac{v+v\cos\theta}{1+v^2\cos\theta/c^2}, \frac{v\sin\theta}{\gamma_v(1+v^2\cos\theta/c^2)}, 0\right) \ , \\
\vec{\mathbf{p}}'_D &= \mathcal{M}(u_D)\left(\frac{v-v\cos\theta}{1-v^2\cos\theta/c^2}, \frac{-v\sin\theta}{\gamma_v(1-v^2\cos\theta/c^2)}, 0\right)
\end{aligned}
$$

where $u_A = 2v/(1+v^2/c^2)$ and $\gamma_v = 1/\sqrt{1-v^2/c^2}$.

Momentum conservation should hold in S' too !

- Momentum conservation in y direction gives

$$\mathcal{M}(u_C)\frac{v\sin\theta}{\gamma_v(1+v^2\cos\theta/c^2)} = \mathcal{M}(u_D)\frac{v\sin\theta}{\gamma_v(1-v^2\cos\theta/c^2)} \ .$$

This leads to

$$\frac{\mathcal{M}(u_C)}{\mathcal{M}(u_D)} = \frac{1+v^2\cos\theta/c^2}{1-v^2\cos\theta/c^2} \ .$$

- Now go to the limit of glancing collision, i.e.

$$\theta = 0 \ , \quad u_C = u_A \ , \quad u_D = 0 \ .$$

In this limit, the above relation yields

$$\frac{\mathcal{M}(u_A)}{\mathcal{M}(0)} = \frac{1+v^2/c^2}{1-v^2/c^2} = \frac{1}{\sqrt{1-u_A^2/c^2}} \ , \qquad (14)$$

the last equality uses $u_A = 2v/(1+v^2/c^2)$.

- Using $\mathcal{M}(0) = m$, we have

$$\mathcal{M}(u_A) = m/\sqrt{1-u_A^2} = \gamma_{u_A} m \ . \qquad (15)$$

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Relativistic momentum and force

- The relativistic momentum is thus

$$\vec{\mathbf{p}} = m\gamma\,\vec{\mathbf{u}}\,. \tag{16}$$

- This definition is the only one possible which allows conservation of momentum to be valid in all frames. Clearly the non-relativistic definition $\vec{\mathbf{p}} = m\vec{\mathbf{u}}$ is not valid at large velocities.
- The relativistic force would naturally be defined as the rate of change of momentum:

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = \frac{d}{dt}(m\gamma\vec{\mathbf{u}})\,. \tag{17}$$

Note that $\vec{\mathbf{F}} \neq m\vec{\mathbf{a}}$, and $\vec{\mathbf{F}} \neq m\gamma\vec{\mathbf{a}}$ either. Indeed in general, force need not even be in the same direction as acceleration.

Problem

Find the conditions under which $\vec{\mathbf{F}}$ and $\vec{\mathbf{a}}$ can be parallel

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Find the conditions under which $\vec{\mathbf{F}}$ and $\vec{\mathbf{a}}$ can be parallel

Outline

Kinetic energy

- Given the momentum $\vec{\mathbf{p}}$, it is possible to define the kinetic energy of a particle as the work done on it while increasing its velocity $\vec{\mathbf{v}}$ from 0 to $\vec{\mathbf{u}}$, in a linear motion.

$$\begin{aligned}
\mathrm{KE} &= \int \frac{d\vec{\mathbf{p}}}{dt} \cdot d\vec{\mathbf{x}} = \int \frac{d}{dt}(m\,\gamma_v\,\vec{\mathbf{v}}) \cdot \vec{\mathbf{v}} dt \\
&= \int \frac{d}{dt}(m\,\gamma_v\,\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) dt - \int m\,\gamma_v\,\vec{\mathbf{v}} \cdot \frac{d\vec{\mathbf{v}}}{dt} dt \\
&= m\,\gamma_u\,u^2 - \frac{m}{2} \int \frac{dv^2}{\sqrt{1 - v^2/c^2}} \\
\mathrm{KE} &= m\,c^2\,(\gamma - 1)\,. \qquad (18)
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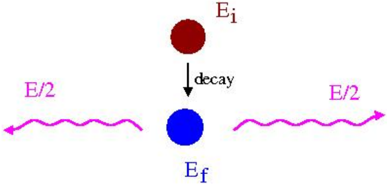
Energy conservation in nuclear decay

The following argument is a variation on Einstein's original one in his $E = mc^2$ paper:
"Does the inertia of a body depend on its energy content ?", Annalen der Physik 18. (1905) 639-641

- Consider a nucleus with mass m_i at rest decaying to another one with mass m_f and two photons leaving in opposite directions with energy $E/2$ each. The conservation of energy gives

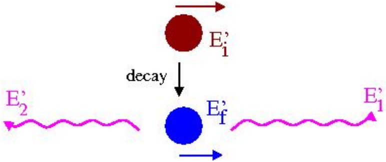
$$E_i = E_f + E \ ,$$

where E_i and E_f are the rest energies of the nuclei. Both are at rest by conservation of momentum.



The same decay in another frame

- Now look at the same decay in a frame where the nucleus is moving with speed v and the two photons are emitted along (or opposite) the line of motion.



- The energy carried by the photons in this frame is $E_1' = (E/2)\gamma(1-\beta)$ and $E_2' = (E/2)\gamma(1+\beta)$, the total photon energy thus being $E' = E\gamma$. The conservation of energy now gives

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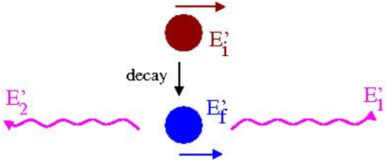

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Postulation of rest mass energy

- The total energy is rest energy plus kinetic energy, i.e.

$$E_i' = E_i + m_i c^2(\gamma - 1)\ , \quad E_f' = E_f + m_f c^2(\gamma - 1)\ .$$

- Combining the above equation with energy conservation relations gives

$$m_i c^2 - m_f c^2 = E\ . \tag{19}$$

- This may be "interpreted" as the nuclei having rest energies equal to $m_i c^2$ and $m_f c^2$, respectively, and the difference being emitted as photons.
- This is thus the "postulation" of rest energy of a particle equal to mc^2, which needs to be confirmed by experiments involving radioactive decays: emitted energy and masses of nuclei. The experiments have confirmed this, and hence we have

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Relativistic momentum and energy

- Relativistic momentum can be uniquely defined using the requirement that conservation of momentum should hold in all frames
- Force need not be parallel to accelaration
- Kinetic energy can be determined from momentum, but the rest mass energy can only be postulated; the identification $E = mc^2$ needs experimental input.