Mathematical Physics, Autumn 2008

Assignment 2 (due Monday 28/9/2008)

1. Let $A = \frac{d}{dx}$ and $B = \frac{d^2}{dx^2}$ be two operators acting on the vector space of cubic polynomials, $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

   (a) Confirm that these are linear operators

   (b) Write down the matrix representations of these operators

   (c) In the matrix representation, calculate the commutator $[A, B]$.

2. Solve the differential equation

$$\frac{d^2}{dx^2} f(x) + m \frac{d}{dx} f(x) = 3x^2$$

   over the vector space of cubic polynomials above, by converting it to a matrix equation, solving the matrix equation, and converting the solution back to a polynomial.

3. Find at least two normalized eigenfunctions, and the corresponding eigenvalues, of the operator

$$\mathcal{O} \equiv \int_0^\pi dy \ \sin(x + y)$$

   which, when acting on $u(y)$, gives

$$f(x) = \int_0^\pi dy \ \sin(x + y) u(y) \ .$$

   [Hint: Try functions of the sinusoidal form.]

4. Consider a linear operator A on a vector space V, such that $A|b\rangle = 0 \Rightarrow |b\rangle = 0$. Show that:

   (a) Every $|a\rangle \in V$ can be uniquely written as $A|c\rangle$ where $|c\rangle \in V$.

   (b) Left inverse of A exists and is given by $A_\ell^{-1}|a\rangle = |c\rangle$.

5. For a normal matrix A,

   (a) Show that if $|x\rangle$ is an eigenvector of A with eigenvalue λ, then it is also an eigenvector of $A^\dagger$ with eigenvalue λ^*.

   (b) Using the above result, show that eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

6. Let matrices A and B have the same eigenvalues, all of which are distinct. Show that:

   (a) All eigenvectors of A are linearly independent

   (b) The matrices A and B are related by a similarity transformation, i.e. $A = S^{-1}BS$.

7. Consider $A = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 5 & 2 \\ 4 & 6 & 1 \end{pmatrix}$.

   (a) Find an LU decomposition of A

   (b) Determine the unique LDU decomposition of A.

   (c) Using either of the above, solve the equation

$$Ax = \begin{pmatrix} 20 & 21 & 19 \end{pmatrix}^T .$$

8. Consider $B = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 5 & 2 \\ 4 & 6 & 1 \\ 1 & 4 & 3 \end{pmatrix}$.

   (a) Find a QR decomposition of B. [Keep any square roots as square roots, do not convert them to real numbers yet.]

   (b) Using the QR decomposition, find the least square fit to the equation

$$Bx = \begin{pmatrix} 20 & 21 & 19 & 17 \end{pmatrix}^T .$$

   [As a final step, convert your answer to real numbers.]

9. Consider the general 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$.

   (a) Find the characteristic polynomial $p(x)$ for A. Show explicitly that A satisfies the equation $p(A) = 0$.

   (b) Show that, if all the "principle subdeterminants" of A are positive, then A is positive definite.

10. Find the characteristic polynomials, minimal polynomials and the Jordan form for the matrices

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

Not to be submitted:

1. Show that all unit 3×3 lower triangular unit matrices form a group under multiplication

2. A rotation in n dimensions is represented by an orthogonal $n \times n$ matrix. Show that any such rotation in 3 dimensions keeps vectors along at least one direction unchanged. For which n can this result be guaranteed ?

3. Consider $A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 1 \\ 3 & 5 & 2 \end{pmatrix}$.

   (a) Find an LUP decomposition of A

   (b) Using either of the above, solve the equation

$$Ax = \begin{pmatrix} 20 & 19 & 21 \end{pmatrix}^T .$$

4. Consider $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

   (a) Show that this matrix is Hermitian, unitary and traceless.

   (b) Using only the above results, determine the eigenvalues of A.

5. Given the characteristic polynomial and the minimal polynomial for an $n \times n$ matrix, is its Jordan form uniquely defined ? Can you prove the above statement for $n = 3, 5, 7$? If not, give a counterexample.