Mathematical Physics, Autumn 2008

Assignment 4 (due Monday 24/11/2008)

1. (a) Find the Cauchy-Riemann conditions in polar form, i.e. find the relationship between $\partial u/\partial r, \partial u/\partial \theta, \partial v/\partial r$ and $\partial v/\partial \theta$ for $f(re^{i\theta}) = u(r,\theta) + v(r,\theta)$ to be differentiable at a point (r,θ).

   (b) Find the Cauchy-Riemann conditions in polar form, when the function is written as $f(re^{i\theta}) = R(r,\theta)e^{i\Theta(r,\theta)}$.

2. (a) Find the region in the complex plane where $0 \leq \mathrm{Im}(e^{z^2}) \leq 1$.

   (b) Find the values of z for which $\mathrm{Log}(\sin z)$ is a real positive number.

3. Consider the integral

$$I \equiv \int_C e^z z^2 dz$$

   where the contour C is $(0,0) \to (1,0) \to (1,1)$. Show that $|I| < 2e\sqrt{2}$.

4. When C is a unit circle traversed anticlockwise, find

$$\oint_C \frac{e^z \sin z}{z^3}$$

5. Find the Laurent series for $\ln(1+z)$. What is its radius of convergence? Does the series converge on the circle of convergence?

6. Find all the poles, their order and the residues for

$$f(z) = \frac{\sin z}{z(z-1/2)^2 \cos(\pi z)}$$

7. Calculate

$$I = \int_0^{2\pi} \frac{d\theta}{a + b\cos^2\theta}$$

8. Calculate $\int_0^\infty e^{ix^2} dx$ by considering the limit of $\int_0^\infty e^{(i-\epsilon)x^2} dx$ as $\epsilon \to 0$, and choosing a proper contour (Hint: use the first octant of the circle of infinite radius).

9. Arfken 7.1.4

10. Arfken 7.2.17
11. Arfken 7.2.18
12. Arfken 7.4.1

Not to be submitted:

1. Show that a function $f(x, y)$ is differentible at z_0 if in a neighbourhood of z_0, the partial derivatives u_x, u_y, v_x, v_y (a) exist (b) are continuous (c) satisfty the Cauchy-Riemann conditions. Point out explicitly in your proof where the conditions (a), (b) and (c) have been used.
2. Show that there is no function analytic in any domain whose real part is $u(x, y) = x^2 + y^2$
3. Show that $|\cos z| \leq \sqrt{|1 - |\sin z|^2|}$
4. Arfken 6.5.11
5. Arfken 7.2.27