Mathematical Methods: Autumn 2008

Final Exam, Tuesday Dec 23, 10:00 am

1. Express the matrix
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$
as the product of a unit lower triangular matrix, a diagonal matrix and and a unit upper triangular matrix.

[5 points]

2. A potential in the $x - y$ plane is given by
$$V = V_0(\cos x \cosh y - \sin y \cosh x) \ .$$
If this is the real part of an analytic function $\Phi(z)$ with $z = x + iy$, find $\Phi(z)$.

[5 points]

3. Find the Laurent series for $f(z) = 1/(z^2 - 1)$ around the point $z = 1$. Specify the region in the complex plane over which the Laurent series equals $f(z)$.

[5 points]

4. A sequence of numbers a_n is generated by the recurrence relation
$$a_n = 4(a_{n-1} - a_{n-2}) \quad (n \geq 2),$$
with $a_0 = 1$ and $a_1 = 3$.

(a) Bring the recurrence relation to a form
$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = M \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix} \ .$$
Evaluate eigenvalues and eigenvectors of the 2×2 matrix M.

(b) Determine the matrix S that will bring M to its Jordan form J through the similarity transformation $S^{-1}MS = J$.

(c) Calculate the appropriate power of M and hence find the general expression for a_n.

[20 points]

continued ⟶

5. Evaluate

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2}dx \ , \quad (a > 0)$$

by choosing appropriate contours in the complex plane and using the residue theorem.

[15 points]

6. We need to evaluate the leading behaviour of the integral

$$I(\omega) \equiv \int_0^\infty e^{i\omega(x - x^2/2)} dx$$

for large ω using the method of steepest descent.

(a) Convert $I(w)$ to an appropriate contour integral, and draw the corresponding contour in the complex plane. The relevant features of the contour near the saddle point should be clearly highlighted.

(b) Obtain the leading behaviour of the above integral for large ω by the steepest descent method.

[15 points]

7. Consider the differential equation

$$z^2 y'' + 2zy' - (2 + z^2)y = 0 \ .$$

(a) Find all the singular points of this equation, and determine the nature of these singular points.

(b) Find two series solutions of this differential equation in the vicinity of $z = 0$. Keep only those terms that are up to $\mathcal{O}(z^3)$ times the leading term in the expansion.

[20 points]

continued ⟶

8. The solutions of the differential equation

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0$$

are Legendre polynomials $P_l(x)$, when l is a positive integer. The first few Legendre polynomials are given by

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2 - 1)/2\ .$$

Given the orthogonality property of Legendre polynomials

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{mn}\ ,$$

find the most general solution to

$$(1-x^2)y'' - 2xy' + 6y = x$$

that is of the form

$$y = \sum_{n=0}^{\infty} a_n P_n(x)\ .$$

[15 points]

— The paper ends —