Mathematical Methods: Autumn 2008
Midterm Exam
Thursday Oct 2, 10:00 am

1. If two matrices A and B have the same eigenvalues, all of which are distinct, show that A and B are related by a similarity transformation. Outline an explicit, step-by-step construction of the transformation matrix.

   [5 points]

2. Find all the solutions of the equation $M^4 = M$ where M is a 4×4 matrix.

   [5 points]

3. An object is acted upon by a force $\vec{F} = \rho\hat{\phi}$ in the circular cylindrical coordinate system.

   (a) Calculate curl F in the circular cylindrical coordinate system.

   (b) Calculate work done by this force while taking the particle along a square of side $2R$, centered at the origin, in the $z = 0$ plane.

   [10 points]

4. Using the QR decomposition, find the least square fit for the following set of simultaneous equations:

$$
\begin{aligned}
x + 4y &= 15 \\
2x + 5y &= 25 \\
2x + 2y &= 10 \\
4x &= 2y
\end{aligned}
$$

   The values of elements of Q, R and the best fit values of x and y should be given numerically, accurate to (at least) two significant figures.

   [15 points]

5. Test the following series for convergence:

$$
s = \sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right)^2
$$

   [10 points]

6. Consider the matrix
$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

   (a) Find the Jordan normal form of A.

   (b) Find the matrix P that would bring A to the Jordan normal form through $P^{-1}AP$. Calculate $P^{-1}AP$ explicitly to check.

   (c) Find the characteristic and minimal polynomials for A.

   [15 points]

7. For the above matrix A, calculate e^A exactly (Hint: Write A as the sum of a diagonal matrix D and a nilpotent matrix N.)

   [10 points]

8. The electromagnetic field tensor is given by ($\alpha, \beta \in \{0, 1, 2, 3\}$)
$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

   The Levi-Civita tensor in four dimensions, the completely antisymmetric tensor of the fourth rank, is defined as

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} 0 & \text{if any two of } \alpha, \beta, \gamma, \delta \text{ are identical} \\ +1 & \text{if } (\alpha, \beta, \gamma, \delta) \text{ is an even permutation of } (0, 1, 2, 3) \\ -1 & \text{if } (\alpha, \beta, \gamma, \delta) \text{ is an odd permutation of } (0, 1, 2, 3) \end{cases}$$

   Explicitly calculate the quantity $\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$. If the angle between $\vec{E}$ and $\vec{B}$ is $45°$ in one reference frame, what are the possible values of this angle in any other reference frame ?

   [15 points]

9. The series expansion of the function
$$f(x) = \sum_1^\infty \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \ldots$$

   converges as n^{-2}. Calculate an "improved" series for $f(x)$ that converges as n^{-3}.

   [15 points]

— [The paper ends] —