Neutrino Physics: Lecture 11

Three neutrino oscillations in constant density matter

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Outline

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The problem in flavour basis

$$i\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = H_f\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}$$

$$H_f = U\, H_{vac}\, U^\dagger + V$$

$$H_{vac} = \begin{pmatrix}0 & 0 & 0\\ 0 & 2\Delta_{21} & 0\\ 0 & 0 & 2\Delta_{31}\end{pmatrix}, \quad V = \begin{pmatrix}V_c & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{aligned} U &= \widetilde{U}_{PMNS} = R_{23}(\theta_{23})U_{13}(\theta_{13},\delta)R_{12}(\theta_{12}) \\ &= \begin{pmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & 0 & s_{13}e^{-i\delta}\\ 0 & 1 & 0\\ -s_{13}e^{i\delta} & 0 & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{pmatrix}\end{aligned}$$

Neutrino conversion probability

Passage through constant matter density

$$P_{\alpha\beta} = \left| \sum_j (U_m)^*_{\beta j} (U_m)_{\alpha j} e^{-i\epsilon_j L} \right|^2$$

$$\begin{pmatrix} \nu_{1m} \\ \nu_{2m} \\ \nu_{3m} \end{pmatrix} = U_m^\dagger \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

$$U_m = \begin{pmatrix} (\nu_{1m})_e & (\nu_{1m})_\mu & (\nu_{1m})_\tau \\ (\nu_{2m})_e & (\nu_{2m})_\mu & (\nu_{2m})_\tau \\ (\nu_{3m})_e & (\nu_{3m})_\mu & (\nu_{3m})_\tau \end{pmatrix}^\dagger$$

Need to calculate

- ϵ_j: Eigenvalues of H_f
- ν_{jm}: Normalized eigenvectors of H_f

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Expanding H_f in small parameters

$$H_f = H_0 + H_1$$

$$H_0 = H_f(\theta_{13} = 0, \Delta_{21} = 0) = \begin{pmatrix} V_c & 0 & 0 \\ 0 & 2s_{23}^2\Delta_{31} & 2s_{23}c_{23}\Delta_{31} \\ 0 & 2s_{23}c_{23}\Delta_{31} & 2c_{23}^2\Delta_{31} \end{pmatrix}$$

H_1: Linear / quadratic terms in two small parameters:

- $\theta_{13} \lesssim 0.2$
- $\alpha \equiv \Delta_{21}/\Delta_{31} \approx 0.03$

Perturbative method:

- Obtain eigenvalues $\epsilon_j^{(0)}$ and eigenvectors $v_j^{(0)}$ of H_0
- Perturbative expansion of ϵ_j and v_j in small parameters

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Eigenvectors and eigenvalues of H_0

$$H_0 = 2\Delta_{31} \begin{pmatrix} \widehat{A} & 0 & 0 \\ 0 & s_{23}^2 & s_{23}c_{23} \\ 0 & s_{23}c_{23} & c_{23}^2 \end{pmatrix}$$

$$\widehat{A} \equiv \frac{V_c}{2\Delta_{31}}$$

Eigenvalues of H_0:

$$\epsilon_1^{(0)} = V_c\ , \quad \epsilon_2^{(0)} = 0\ , \quad \epsilon_3^{(0)} = \Delta_{31}$$

Eigenvectors of H_0:

$$v_1^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2^{(0)} = \begin{pmatrix} 0 \\ c_{23} \\ -s_{23} \end{pmatrix}, \quad v_3^{(0)} = \begin{pmatrix} 0 \\ s_{23} \\ c_{23} \end{pmatrix}$$

Higher order corrections to ϵ_j and v_j

First order:

$$\begin{aligned}\epsilon_j^{(1)} &= \langle v_j^{(0)}|H_1|v_j^{(0)}\rangle \\ |v_j^{(1)}\rangle &= \sum_{k\neq j} |v_k^{(0)}\rangle \frac{\langle v_k^{(0)}|H_1|v_j^{(0)}\rangle}{\epsilon_j^{(0)} - \epsilon_k^{(0)}}\end{aligned}$$

Second order:

$$\epsilon_j^{(2)} = \langle v_j^{(0)}|H_1|v_j^{(1)}\rangle$$

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Second order:

$$\epsilon_j^{(2)} = \langle v_j^{(0)} | H_1 | v_j^{(1)} \rangle$$

✓

Perturbatively calculated conversion probability

$$\epsilon_j = \epsilon_j^{(0)} + \epsilon_j^{(1)} + \epsilon_j^{(2)} + \ldots \quad ✓$$

$$v_j = v_j^{(0)} + v_j^{(1)} + \ldots \quad ✓$$

$$U_m^\dagger = \begin{pmatrix} (\nu_{1m})_1 & (\nu_{1m})_2 & (\nu_{1m})_3 \\ (\nu_{2m})_1 & (\nu_{2m})_2 & (\nu_{2m})_3 \\ (\nu_{3m})_1 & (\nu_{3m})_2 & (\nu_{3m})_3 \end{pmatrix}$$

$$\nu_{jm} = v_j / |v_j|$$

Net conversion probability

$$P_{\alpha\beta} = \left| \sum_j (U_m)^*_{\beta j} (U_m)_{\alpha j} e^{-i\epsilon_j L} \right|^2$$

Perturbatively calculated conversion probability

$$\epsilon_j = \epsilon_j^{(0)} + \epsilon_j^{(1)} + \epsilon_j^{(2)} + \ldots$$
$$v_j = v_j^{(0)} + v_j^{(1)} + \ldots$$

$\nu_{jm} \rightarrow$ normalized eigenvector

$$U_m^\dagger = \begin{pmatrix} (\nu_{1m})_1 & (\nu_{1m})_2 & (\nu_{1m})_3 \\ (\nu_{2m})_1 & (\nu_{2m})_2 & (\nu_{2m})_3 \\ (\nu_{3m})_1 & (\nu_{3m})_2 & (\nu_{3m})_3 \end{pmatrix}$$

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Conversion proability and CP violation

$$\alpha \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{31}}, \; \Delta \equiv \frac{\Delta m^2_{31} L}{4E}, \; \hat{A} \equiv \frac{2\sqrt{2} G_F n_e E}{\Delta m^2_{31}}$$

$\frac{V_c}{2\Delta_{31}}$

$$\begin{aligned} P_{e\mu} \simeq\; & \sin^2 2\theta_{13} \sin^2\theta_{23} \frac{\sin^2[(1-\hat{A})\Delta]}{(1-\hat{A})^2} \\ +\; & \alpha \sin 2\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin\delta_{CP} \sin(\Delta) \frac{\sin(\hat{A}\Delta)}{\hat{A}} \frac{\sin[(1-\hat{A})\Delta]}{(1-\hat{A})} \\ +\; & \alpha \sin 2\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos\delta_{CP} \cos(\Delta) \frac{\sin(\hat{A}\Delta)}{\hat{A}} \frac{\sin[(1-\hat{A})\Delta]}{(1-\hat{A})} \\ +\; & \alpha^2 \cos^2\theta_{23} \sin^2 2\theta_{12} \frac{\sin^2(\hat{A}\Delta)}{\hat{A}^2} \end{aligned}$$

(Cervera et al. 2000; Freund, Huber, Lindner, 2000; Huber, Winter, 2003; Akhmedov et al, 2004)

$\Delta \equiv \Delta_{31} L$ (Slide from W. Winter)

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- Antineutrinos: $P_{\bar{e}\bar{\mu}} = P_{e\mu}(\delta_{\mathrm{CP}}, \rightarrow -\delta_{\mathrm{CP}}, \hat{A} \rightarrow -\hat{A})$

(Cervera et al. 2000; Freund, Huber, Lindner, 2000; Huber, Winter, 2003; Akhmedov et al, 2004)

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\end{aligned}$$

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- **Magic baseline:** $\sin(\hat{A}\Delta) = 0 \Rightarrow \sqrt{2} G_F n_e(L) L = 2\pi$

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- **Silver:** $P_{e\tau} = P_{e\mu}(s^2_{23} \leftrightarrow c^2_{23}, \sin 2\theta_{23} \rightarrow -\sin 2\theta_{23})$

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Conversion proability and CP violation

$$P_{e\mu} \simeq \sin^2 2\theta_{13} \begin{smallmatrix}\cos^2\theta_{23} \\ \sin^2\theta_{23}\end{smallmatrix} \frac{\sin^2[(1 \pm \hat{A})\Delta]}{(1 \pm \hat{A})^2}$$

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$$+ \; \alpha^2 \begin{smallmatrix}\sin^2\theta_{23} \\ \cos^2\theta_{23}\end{smallmatrix} \sin^2 2\theta_{12} \frac{\sin^2(\hat{A}\Delta)}{\hat{A}^2}$$

- **Antineutrinos:** $P_{\bar{e}\bar{\mu}} = P_{e\mu}(\delta_{CP}, \rightarrow -\delta_{CP}, \hat{A} \rightarrow -\hat{A})$
- **Magic baseline:** $\sin(\hat{A}\Delta) = 0 \Rightarrow \sqrt{2} G_F n_e(L) L = 2\pi$
- **Silver:** $P_{e\tau} = P_{e\mu}(s_{23}^2 \leftrightarrow c_{23}^2, \sin 2\theta_{23} \rightarrow -\sin 2\theta_{23})$
- **Platinum, Superb.:** $P_{\mu e} = P_{e\mu}(\delta_{CP}, \rightarrow -\delta_{CP})$

(Cervera et al. 2000; Freund, Huber, Lindner, 2000; Huber, Winter, 2003; Akhmedov et al, 2004)

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appearance expts

Outline

Long baseline experiments

- Useful when $\Delta_{31} \sim V_C \sim 10^{-13}$ eV
- Corresponding energy $E \sim 1$–10 GeV
- $\Delta_{31} L \sim 1 \Rightarrow L \sim 1000$–$10000$ km

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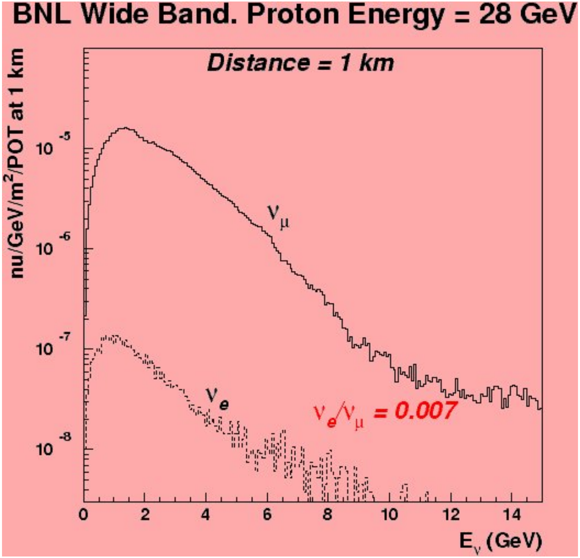
Superbeam experiments



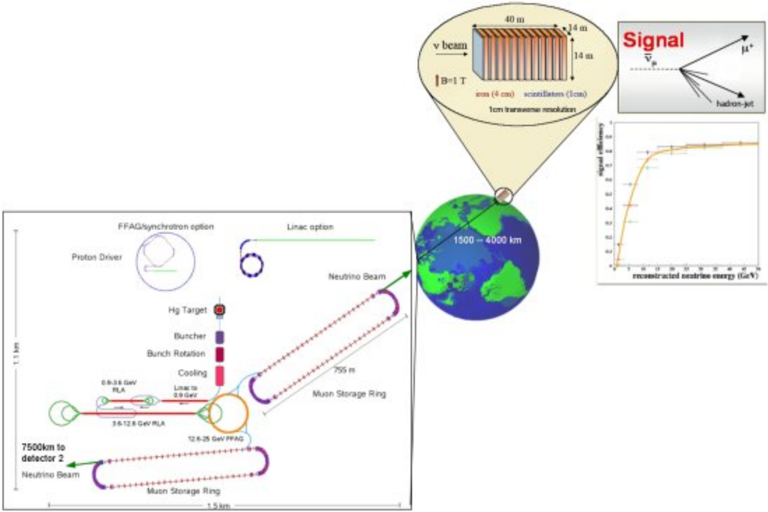
Conventional beam

$P_{\nu_\mu \to \nu_e}$

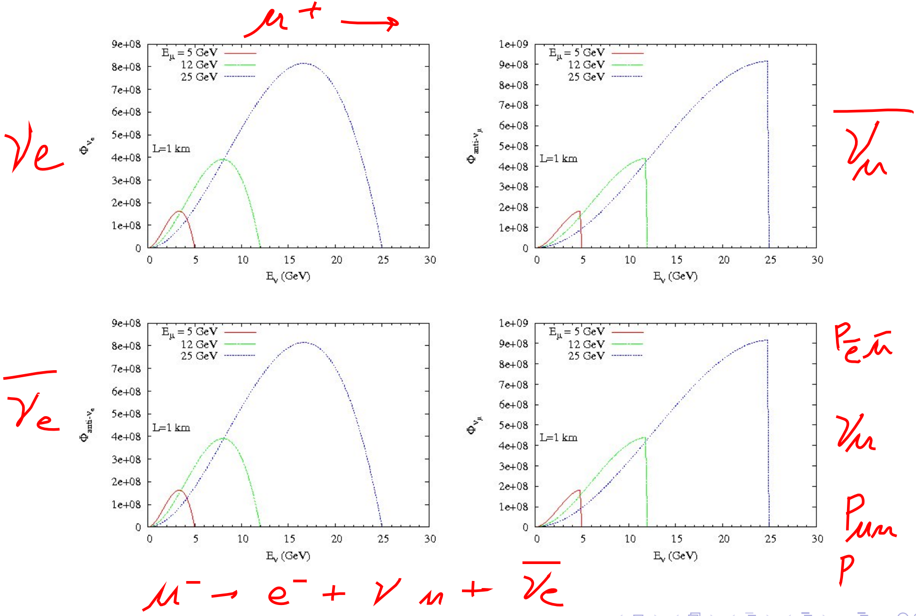
beam background ↓

known technology ↑

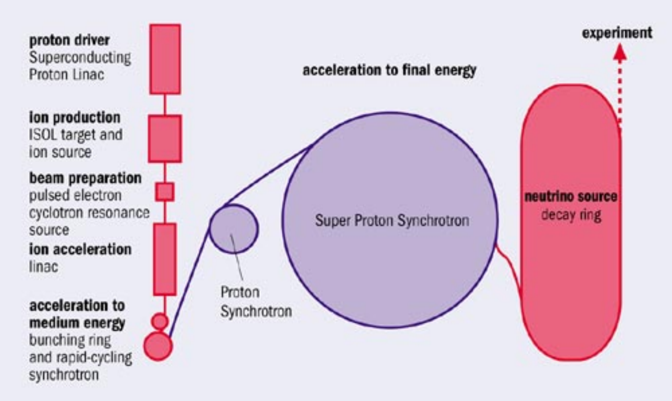
Neutrino factory / muon storage ring



Neutrino factory: neutrino spectra

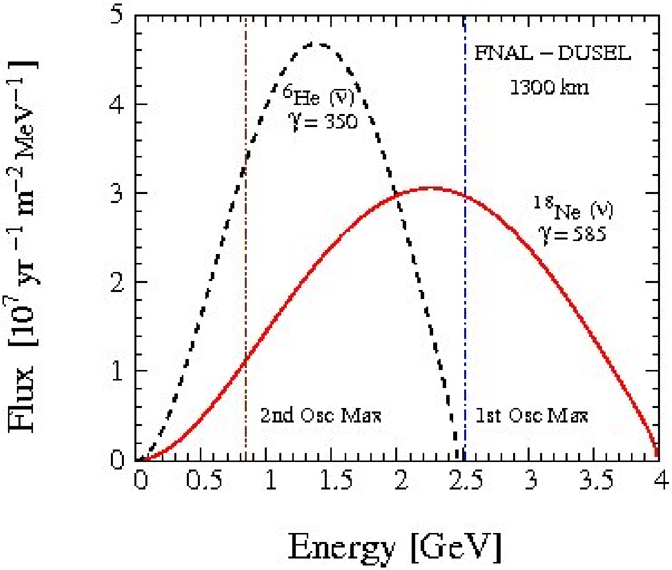


Beta beam



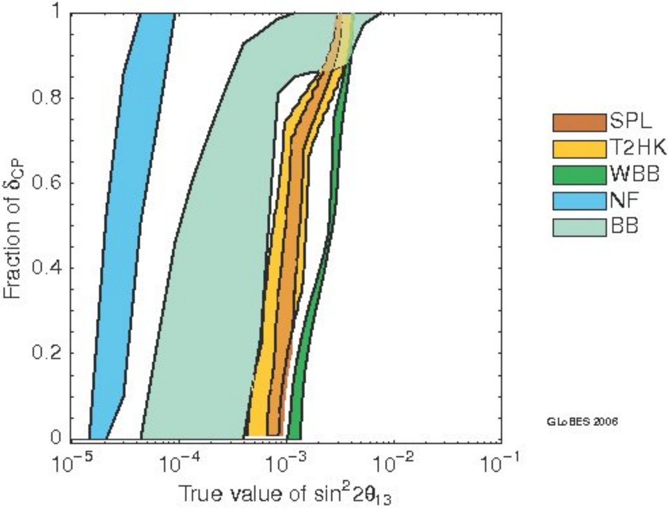
	Ion	τ(s)	E_0 (MeV)
ν_e	${}^{18}_{10}\mathrm{Ne} \to {}^{18}_{9}\mathrm{F}$	2.41	3.92
$\bar{\nu}_e$	${}^{6}_{2}\mathrm{He} \to {}^{6}_{3}\mathrm{Li}$	1.17	4.02

Beta beam

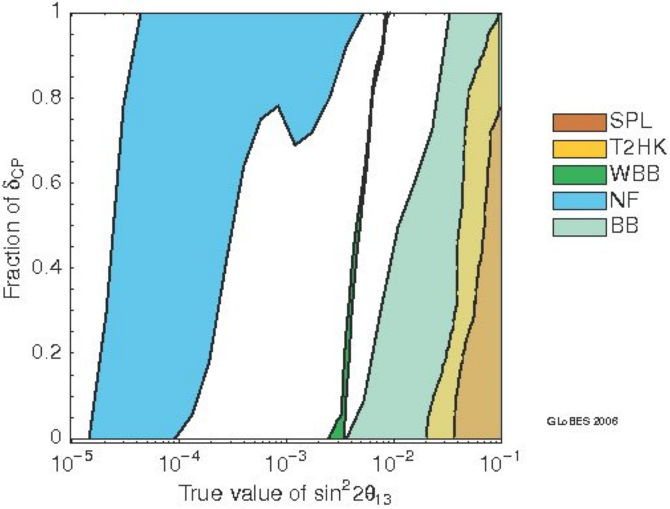


FNAL – DUSEL
1300 km
^{6}He ($\bar{\nu}$)
γ = 350
^{18}Ne (ν)
γ = 585
2nd Osc Max
1st Osc Max
Flux [10^7 yr^{-1} m^{-2} MeV^{-1}]
Energy [GeV]
0
1
2
3
4
5
0
0.5
1
1.5
2
2.5
3
3.5
4

The future of θ_{13} measurement



The future of mass ordering (hierarchy) measurement



The future of *CP* violation measurement