Quantum Mechanics 2, Spring 2015

Assignment #1, Due 17/02/2015 in class

1. Show that if P is a projection operator ($P^2 = P = P^\dagger$), then (I - 2P) is both Hermitian and unitary. Is the converse true (is every operator that is both Hermitian and unitary of this form) ?

2. Show that it is impossible to have two Hermitian operators P, Q in a Hilbert space that satisfy $[P, Q] = i\hbar I$. (Hint: Start by calculating $[P, Q^n]$ for an arbitrary integer n, and show that $\langle P \rangle \langle Q \rangle$ is unbounded.) How is this compatible with the position-momentum uncertainty relation ?

3. Given creation and annihilation operators

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad \text{and} \quad a|n\rangle = \sqrt{n}|n-1\rangle \ ,$$

   calculate
   (a) $[a, a^\dagger]$ and $\{a, a^\dagger\}$
   (b) $\langle 0|aa^\dagger aaa^\dagger a^\dagger|0\rangle$
   (c) $\langle m|a^\dagger a^\dagger a^\dagger a|n\rangle$

4. Given a one dimensional potential well

$$V(x) = -|V_0(x-a)| \quad \text{for} \quad -a < x < a, \quad \text{zero otherwise} \ .$$

   (a) Draw qualitatively (no need to solve the Schrödinger equation) the lowest three bound state wavefunctions. Assume that the well is deep enough. A free hand sketch is sufficient.
   (b) Draw qualitatively two wavefunctions with $E > 0$, one with $E \ll |V_0|$, one with $E \gg |V_0|$.

5. Find the relation between $\Delta x|\alpha\rangle$ and $\Delta p|\alpha\rangle$ that satisfies the *equality*

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4 \ .$$

   Solve this differential equation and find $|\alpha\rangle$.

6. Calculate the expectation value of the operator $X = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ in the following three systems:
(a) all particles in the state $|A\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$
(b) Half the particles in the state $|B_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and half in $|B_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
(c) all particles in the state $|C\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

7. The Pauli σ matrices, that come handy in many problems involving rotation, are given by

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

(a) Show that $(\vec{\sigma}.\hat{n})^2 = 1$ for every unit vector $\hat{n}$
(b) Using the above, show that $e^{i\theta\vec{\sigma}.\hat{n}} = \cos\theta + i\vec{\sigma}.\hat{n}\sin\theta$

8. K-mesons are created as "strangeness" eigenstates $|K\rangle$ or $|\overline{K}\rangle$. Since the strangeness operator does not commute with the Hamiltonian, these are not the mass eigenstates. The mass eigenstates are

$$|K_1\rangle \equiv \frac{1}{\sqrt{2}}(|K\rangle + |\overline{K}\rangle) \quad \text{and} \quad |K_2\rangle \equiv \frac{1}{\sqrt{2}}(|K\rangle - |\overline{K}\rangle) .$$

The Hamiltonian in the mass eigenstate basis is

$$H = \begin{pmatrix} m_1 - i\Gamma_1/2 & 0 \\ 0 & m_2 - i\Gamma_2/2 \end{pmatrix} .$$

If a $|K\rangle$ meson is produced at $t = 0$, calculate the probability that a $|\overline{K}\rangle$ meson is observed at time t. (These are the $K - \overline{K}$ oscillations that led to the first signature of CP violation.)