Quantum Mechanics 2, Spring 2015

Assignment #2, Due 17/3/15

1. Consider a simple harmonic oscillator. Given

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \ , \quad p = i\sqrt{\frac{m\hbar\omega}{2}}(-a + a^\dagger) \ ,$$

   (a) Calculate $\langle n'|x|n\rangle$ and $\langle n'|p|n\rangle$.

   (b) Obtain the ground state $\langle x'|0\rangle$ by solving for $a|0\rangle = 0$.

   (c) Similarly, using $a|1\rangle \propto |0\rangle$ etc, obtain the first excited state $\langle x'|1\rangle$.

   (d) Calculate the "correlation function" $C(t) = \langle x(t)x(0)\rangle$ for the ground state, where $x(t)$ is the position operator in the Heisenberg picture.

2. Consider the motion of an electron in a uniform magnetic field along the z axis. Choose the gauge $\vec{A} = (0, Bx, 0)$.

   (a) Argue that the wavefunction may be written in the form

$$\langle \vec{x}|\psi\rangle = \exp(ik_z z + ik_y y)\phi(x)$$

   (b) Show that the Schrödinger's equation may be written in the form

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}\phi(x) + \frac{e^2B^2}{2mc^2}(x - x_0)^2\phi(x)\right) = E_{xy}\phi(x)$$

   where $E_{xy} = E^2 - \hbar^2 k_z^2$. Determine x_0.

   (c) Looking at the above equation as a simple harmonic oscillator, determine the eigenvalues of the original Hamiltonian.

   (d) Interprete the solution $\langle \vec{x}|\psi\rangle$ in terms of the classical motion of an electron in a uniform magnetic field.

3. Consider the Aharonov-Bohm experiment (fig. 2.8 in Sakurai). Electrons with momentum p impinge on the double-slit at a constant rate from one side. Let the distance between the slits be d.

   (a) Find the interference pattern measured by detectors kept at a distance $L \gg d$ on the other side of the screen, in the absence of any magnetic field in the solenoid. What is the distance between successive maxima ?

   (b) Consider a detector that is situated symmetrically with respect to the two slits at $L \gg d$. Turning on the magnetic field in the solenoid will change the rate of particles detected. Find the values of the magnetic flux for which the rate will be a maximum.

4. Using the density matrix formalism,

   (a) Show that a unitary evolution cannot take a pure state to a mixed state or the other way around.

   (b) Can a mixed state change into another mixed state with unitary evolution ? Give proof / counterexample.

   (c) Consider an ensemble of spin-1 particles. In the basis of $(|+1\rangle, |0\rangle, |-1\rangle)$ where the basis states are labelled by their J_z values, the density matrix is given by

$$\rho = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} .$$

   Calculate the ensemble averages $[J_z], [J_z^2], [J_x], [J_x^2]$.

   (d) Find any mixture of states (each of which is a linear superposition of $|+1\rangle, |0\rangle, |-1\rangle$) which will give rise to the above density matrix.

Recommended problems (not for submission):

1. Show that

$$[x_i, g(\vec{p})] = i\hbar \frac{\partial g}{\partial p_i} , \quad [p_i, f(\vec{x})] = -i\hbar \frac{\partial f}{\partial x_i} .$$

2. Normalize the transition amplitude

$$\langle x_n t_n | x_{n-1} t_{n-1} \rangle = \mathcal{N} \exp \left[\frac{iS(n, n-1)}{\hbar} \right]$$

   using the condition $\langle x'' t'' | x' t' \rangle |_{t'' \to t'} = \delta(x'' - x')$. Further, show that this leads to

$$i\hbar \frac{\partial}{\partial t} \langle xt | x_1 t_1 \rangle = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \langle xt | x_1 t_1 \rangle + V \langle xt | x_1 t_1 \rangle .$$

3. Check that the electromagnetic probability flux

$$\vec{j} = \left(\frac{\hbar}{m} \right) \mathrm{Im}(\psi^* \vec{\nabla} \psi) - \left(\frac{e}{mc} \right) \vec{A} |\psi|^2$$

   satisfies the continuity equation in the presence of the potential $\vec{A}$.

4. In the class, we saw that for a thermal ensemble,

$$\rho_{kk} = e^{-\beta E_k} / \sum e^{-\beta E_k}$$

   where β is a Lagrange multiplier. Using the classical limit of $[E]$ for a gas at temperature T, determine β in terms of T.