

Quantum Mechanics 2, Spring 2015

Assignment #4, Due 30/04/2015

1. Consider an α particle with $E > 0$ in a nucleus, under the influence of a short-distance attractive potential V_0 and the Coulomb repulsion: $V(x) = -V_0 \Theta(R - x) + Z_1 Z_2 e^2/x$. Calculate the probability for it to tunnel out as a function of its energy in the limit $E \ll Z_1 Z_2 e^2/R$.

The mean lifetime for α decay is $\tau = (2R/v_i)|T(E)|^{-2}$ where $v \sim 10^9$ cm/s is the typical speed of the particle inside the nucleus. Use $R \sim 10^{-12}$ cm and estimate the lifetime (in years) of a $Z_1 = 90$ nucleus that decays with $E_\alpha = 5$ MeV. (An order of magnitude estimation is fine.)

2. Calculate the effect of a finite nuclear size on the energy of H atom states with $n = 1$ and $n = 2$: Take the H nucleus to be a sphere of radius 1 fm with uniform charge density, and calculate $\Delta E/E$ numerically for all the states.
3. (a) Estimate the ranges of magnetic field (numerically, in gauss) for which the $L \cdot S$ term, the $(L + 2S) \cdot B$ term and the $|B|^2$ term respectively (from the Hamiltonian for an electron in a Hydrogen atom) dominate over the others.
(b) Consider the 3d levels of an electron in the Hydrogen atom. Show the energy level diagrams (energy eigenstates as functions of $|B|$) in the limits of weak and strong magnetic field (strong magnetic field still not strong enough to take into account the $|B|^2$ term). Indicate the values of all level splittings in terms of

$$a \equiv \frac{e\hbar|B|}{2mc} \quad \text{and} \quad b_j \equiv \frac{1}{2m^2c^2} \left\langle \frac{1}{r} \frac{dV}{dr} \right\rangle_j$$

4. An Hydrogen atom in $n = 2$ state is kept in mutually perpendicular constant electric and magnetic fields. For strong fields (so that the spin-orbit interaction may be neglected), determine the energy level splittings. Draw the energy level diagram.
5. Using the trial function $|\alpha\rangle = e^{-ar^2}$, find an upper bound on the ground state energy of
 - (a) a simple Harmonic oscillator in one dimension
 - (b) an electron in an Hydrogen atom

In both the cases, estimate how close the ground state wavefunction has been guessed by computing a lower bound on $|\langle \tilde{\alpha}|0\rangle|^2$ (where $|\tilde{\alpha}\rangle$ is normalized $|\alpha\rangle$). (You may use the knowledge of the exact energy spectra, but do not use the actual forms of $|0\rangle$ even if you know it).

6. For a H_2 molecule, let the interatomic interactions be taken as perturbations, so that the unperturbed ground state is $U_0 = U_{100}(\vec{r}_1)U_{100}(\vec{r}_2)$. (See Fig. 5.3 in Sakurai). The perturbation may be expanded in powers of r_i/r to get

$$V = \frac{e^2}{r^3}(x_1x_2 + y_1y_2 - 2z_1z_2) + \mathcal{O}\left(\frac{1}{r^4}\right)$$

(You may try to derive this, but no need to show it here.)

- (a) Calculate the lowest order correction to the ground state energy. (Assume that the $\mathcal{O}(1/r^4)$ terms above give no first order contribution). The answer may be in the form of a summation. Find a lower bound on this summation, and hence on the ground state energy.
- (b) Choose the trial function $|\alpha\rangle = U_{100}(\vec{r}_1)U_{100}(\vec{r}_2)(1 + aV)$ to determine an upper bound on the ground state energy.

Recommended:

Problems 1 – 21 from Sakurai chapter 5. An extremely good collection.