# Quantum Mechanics 2, Spring 2015

Final Exam, 17/5/2015

1. An operator $A$ satisfies the following commutation relations:

$$[A, J_z] = A \ , \qquad [\,[A, J^2],\ J^2] = 2(AJ^2 + J^2 A) \ .$$

Find the conditions for which transitions from $|jm\rangle$ to $|j'm'\rangle$ are possible when mediated by the operator $A$.

[10 points]

2. Determine whether an attractive potential in one dimension,

$$V(x) = -|V_0| e^{-\alpha^2 x^2} \ ,$$

always has a bound state solution, however small $V_0$ may be. A complete logical argument is expected.

[10 points]

3. Consider a one-dimensional "deep" potential well,

$$V(x) = -|V_0| \left(1 - \left|\frac{x}{a}\right|\right) \quad \text{for} \quad |x| < a \ , \qquad V(x) = 0 \quad \text{for} \quad |x| > a \ .$$

Estimate the density of bound states $\rho(E)$ for $|E| \ll |V_0|$. State the conditions for the validity of the approximation used.

[10 points]

4. Consider the Hamiltonian represented by

$$H = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & a \\ a^* & a^* & b \end{pmatrix} \ ,$$

where $|a| \ll |b|$. Formulate this as a problem in perturbation theory, and calculate the eigenvalues and eigenstates accurate to the leading order in $|a/b|$.

[10 points]

5. A tritium nucleus at rest undergoes the spontaneous beta decay

$$^3\mathrm{H} \rightarrow \ ^3\mathrm{He}^+ + e^- + \bar{\nu}_e \ .$$

The emitted (electron + antineutrino) energy equals 17 keV. Assume that the electron in the tritium atom was initially in its ground state. Calculate the probability (numerical value expected) that the electron of the newly formed helium ion is also in its ground state. Justify all your approximations with numerical estimates. (You may use the information in the appendix of Sakurai.)

[10 points]

6. Consider an ensemble of particles of mass $m$ in the ground state of an attractive one-dimensional delta function potential: $V_0(x) = -\beta\delta(x)$. These particles may be ionized by applying an additional harmonic potential $V(x,t) = eEx\cos(\omega t)$, where $\hbar\omega > E_0$ (binding energy of the ground state). Estimate the ionization rate through the following steps:

   (a) Find the ground state energy and the normalized wavefunction of the delta function potential.

   [10 points]

   (b) Consider a state $|k\rangle$ with wavenumber $k$, normalized as $\langle x|k\rangle = e^{ikx}/\sqrt{L}$, where $L$ is an arbitrarily large "box size". Calculate the matrix element $\langle k|V|0\rangle$. Hence determine the probability of transition to the state $|k\rangle$.

   [10 points]

   (c) Using the density of states with wavenumber $k$, estimate the total ionization rate. What is the limit of large box size ?

   [10 points]

7. Consider the scattering of a plane wave with wavenumber $k$ from a spherically symmetric rectangular potential barrier,

$$V(r) = V_0 \quad \text{for} \quad r < R \ , \qquad \text{zero otherwise.}$$

   (a) Calculate the scattering amplitude in the first order Born approximation (for all energies). What is the maximum value of $V_0$ for which this approximation will be valid ?

   [10 points]

   (b) Determine the total cross section at small energies ($kR \ll 1$). Comment on its dependence on $k$ and $R$.

   [10 points]