Quantum Mechanics 2, Spring 2015

Midterm #1, Apr 5, 2015

1. If $\mathcal{P}$ and $\mathcal{X}$ are the momentum and position operators in one dimension, and the Hamiltonian is

$$H = \frac{[\mathcal{P} - (e/c)\mathcal{A}(\mathcal{X})]^2}{2m} + \frac{1}{2} m\omega^2 \mathcal{X}^2 + C\mathcal{X}^4 \quad ,$$

Determine the equations of motion for the Heisenberg operators $\mathcal{P}(t)$ and $\mathcal{X}(t)$. (No need to solve these equations.)

[10]

2. Consider a one-dimensional simple harmonic oscillator. The annihilation and creation operators are

$$\left.\begin{matrix} a \\ a^\dagger \end{matrix}\right\} = \sqrt{\frac{m\omega}{2\hbar}} \left(x \pm \frac{ip}{m\omega} \right) \ , \text{ with } a|n\rangle = \sqrt{n}|n{-}1\rangle \ , a^\dagger|n\rangle = \sqrt{n+1}|n{+}1\rangle \ .$$

Using the above (without using the explicit forms for wavefunctions),

(a) Construct a state $|\alpha\rangle = c_0|0\rangle + c_1|1\rangle$ such that $\langle x\rangle$ is a maximum.

(b) For the state $|\alpha\rangle$, determine the uncertainty product $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$.

[5+15]

3. The total angular momentum of a spin-S particle with orbital angular momentum L is $\vec{J} = \vec{L} + \vec{S}$. The interactions between these angular momenta give rise to a term $H_{\rm int} = a\vec{L}\cdot\vec{S}$ in the Hamiltonian.

(a) Which of the quantum numbers $j, m_j, \ell, m_\ell, s, m_s$ are conserved ? Show explicitly by computing the relevant commutators. (Explicit arguments necessary for each of the quantities.)
(You may find the following relations useful:
$\vec{L}\cdot\vec{S} = L_z S_z + (L_+ S_- + L_- S_+)/2, \quad [J_z, J_\pm] = \pm\hbar J_\pm$)

(b) Calculate the eigenvalues of $H_{\rm int}$ in terms of the conserved quantum numbers.

[15+5]

4. A particle A is known to have spin 1/2. When it scatters on a target, an initial pure state $|i\rangle$ changes to the final state $|f\rangle$ such that $|f\rangle = S|i\rangle$, where the "S-matrix" is $S = a + b\sigma_z$. (Here z axis is some fixed direction.)

   (a) Derive the condition on a and b that would ensure the conservation of number of particles of A. (Explicit reasoning expected.)

   (b) The polarization of an ensemble is defined as the ensemble average of $\vec{\sigma}$, i.e. $\vec{P} \equiv [\vec{\sigma}]$. If one starts with an unpolarized ($\vec{P}_i = \vec{0}$) beam of A particles, calculate the density matrix of the beam after scattering (in terms of a and b). Hence, determine the polarization $\vec{P}_f$ of the beam after scattering.

   (c) This polarized ensemble of particles then decays. Each A particle decays into another spin-1/2 particle B and a positive helicity photon. Find the angular distribution of the outcoming photons in the rest frame of the A particles (in terms of $\vec{P}_f$).

[5+10+10]

5. Consider an electron in the 3d atomic orbital, with $m = +1/2$ and $m_s = +1/2$.

   (a) Find the probability that this electron has an angular momentum quantum number $j = 3/2$.

   (b) This electron (in the $|j = 3/2, m = +1/2\rangle$ state) can jump to a state $|n = 2, j', m'\rangle$ through a dipole moment interaction term, $V_{\text{dipole}} \equiv \rho\, x$. List the values of $|j', m'\rangle$ to which such a transition is possible.

   (c) Calculate the ratio of intensities of the transitions of this electron (in the $|j = 3/2, m = +1/2\rangle$ state) to states of the form $|n = 2, j' = 3/2, m'\rangle$, for different m' values.

[5+10+10]