Quantum Mechanics 2, Spring 2016

Assignment #3, Due 22/04/2016

[You will need to use numerical and plotting software like Mathematica to solve many of these problems.]

1. Consider an α particle with $E > 0$ in a nucleus, under the influence of a short-distance attractive potential of magnitude V_0, and the Coulomb repulsion:
   $V(x) = -V_0\, \Theta(R - x) + Z_1 Z_2 e^2/x$. Calculate the probability for it to tunnel out as a function of its energy in the limit $E \ll Z_1 Z_2 e^2/R$.

   The mean lifetime for α decay is $\tau = (2R/v_i)|T(E)|^{-2}$ where $T(E)$ is the tunnelling amplitude and $v \sim 10^9$ cm/s is the typical speed of the particle inside the nucleus. Use $R \sim 10^{-12}$ cm and estimate the lifetime (in years) of a $Z_1 = 90$ nucleus that decays with $E_\alpha = 5$ MeV. (An order of magnitude estimation is fine.)

2. Using the trial function $|\alpha\rangle = e^{-ar^2}$, find an upper bound on the ground state energy of

   (a) a simple Harmonic oscillator in one dimension

   (b) an electron in an Hydrogen atom

   In both the cases, estimate how close the ground state wavefunction has been guessed by computing a lower bound on $|\langle\tilde{\alpha}|0\rangle|^2$ (where $|\tilde{\alpha}\rangle$ is normalized $|\alpha\rangle$). (You may use the knowledge of the exact energy spectra, but do not use the actual forms of $|0\rangle$ even if you know it).

3. Consider one-dimensional potentials of the form $V = ax^n$. Using WKB approximation, find the density of states dN/dE as a function of energy E. Find n such that the levels are equispaced.

4. Consider an anharmonic oscilator corresponding to the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 + bx^4 \ .$$

   (a) To first order in perturbation theory, calculate the energy shift of the n^{th} level of the corresponding harmonic oscillator.

   (b) Calculate the first order corrections to the ground sate and the first excited state.

   (c) Plot the unperturbed wavefunction for the ground state, and the first-order-improved wavefunction. Choose appropriate values of parameters such the perturbation theory is valid and the features of first-order corrections are clearly visible.

   (d) Comment on the features of the plots.

5. Calculate the effect of a finite nuclear size on the energy of H atom states with $n = 1$ and $n = 2$:

   (a) Take the H nucleus to be a sphere of radius 1 fm with uniform charge density, and calculate $\Delta E/E$ numerically for all the states.

   (b) Plot the unperturbed radial wavefunctions for these states, and the first-order-improved wavefunctions.

   (c) Comment on the features of the plots.

6. (a) Estimate the ranges of magnetic field (numerically, in gauss) for which the $L \cdot S$ term, the $(L + 2S) \cdot B$ term and the $|B|^2$ term respectively (from the Hamiltonian for an electron in a Hydrogen atom) dominate over the others.

   (b) Consider the 3d levels of an electron in the Hydrogen atom. Show the energy level diagrams (energy eigenstates as functions of $|B|$) in the limits of weak and strong magnetic field (strong magnetic field still not strong enough to take into account the $|B|^2$ term). Indicate the values of all level splittings in terms of

$$a \equiv \frac{e\hbar|B|}{2mc} \quad \text{and} \quad b_j \equiv \frac{1}{2m^2c^2}\left\langle \frac{1}{r}\frac{dV}{dr} \right\rangle_j$$

7. An Hydrogen atom in $n = 2$ state is kept in mutually perpendicular constant electric and magnetic fields. For strong fields (so that the spin-orbit interaction may be neglected), determine the energy level splittings. Draw the energy level diagram.

8. For a H_2 molecule, let the interatomic interactions be taken as perturbations, so that the unperturbed ground state is $U_0 = U_{100}(\vec{r_1})U_{100}(\vec{r_2})$. (See Fig. 5.3 in Sakurai). The perturbation may be expanded in powers of r_i/r to get

$$V = \frac{e^2}{r^3}(x_1x_2 + y_1y_2 - 2z_1z_2) + \mathcal{O}\left(\frac{1}{r^4}\right)$$

   (You may try to derive this, but no need to show it here.)

   (a) Calculate the lowest order correction to the ground state energy. (Assume that the $\mathcal{O}(1/r^4)$ terms above give no first order contribution). The answer may be in the form of a summation. Find a lower bound on this summation, and hence on the ground state energy.

   (b) Choose the trial function $|\alpha\rangle = U_{100}(\vec{r_1})U_{100}(\vec{r_2})(1 + aV)$ to determine an upper bound on the ground state energy.

Recommended:
Problems 1 – 21 from Sakurai chapter 5. An extremely good collection.