Quantum Mechanics 2, Spring 2016

Midterm, Apr 17, 2016

Total marks: 100

1. Consider a spin-1 particle in a magnetic field $\vec{B} = B_0 \hat{z}$. For the following problems, use the basis $|j\ j_z\rangle = \{|1\ +1\rangle, |1\ 0\rangle, |1\ -1\rangle\}$.

   (a) Starting from the operations of $J_{\pm,z}$ on the basis vectors, determine the operators J_x, J_y, J_z.

   [10]

   (b) If the system initially consists of half the particles in the $j_z = +1$ state and half the particles in the $j_x = +1$ state, write down its initial density matrix in the $|j, j_z\rangle$ basis.

   [10]

   (c) The interaction Hamiltonian is $H_{\text{int}} = a\vec{J} \cdot \vec{B}$. Write down the differential equations for the evolutions of the elements of the density matrix ρ_{ij}, and solve them.

   [10]

   (d) Hence calculate the ensemble average $[J_x]$ as a function of time. Interpret the result.

   [10]

   [You may use $\qquad J_\pm |j\ m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}|j\ m \pm 1\rangle \qquad$.]

2. Consider the Hamiltonian

$$H = \begin{pmatrix} A & c & d \\ c & B & f \\ d & f & B \end{pmatrix},$$

   where $c, d, f \ll A, B$.

   (a) Calculate the eigenstates of this Hamiltonian correct to first order in perturbation theory. Clearly state the conditions for validity of the perturbative expansion.

   [10]

   (b) Calculate the eigenvalues of this Hamiltonian correct to second order in perturbation theory. Clearly state the conditions for validity of the perturbative expansion.

   [10]

3. Consider an electron in the $n = 2$ energy level of a Hydrogen atom. In the presence of a magnetic field $B\hat{z}$, the net Hamiltonian may be written as

$$H = H_0 + V \ , \qquad V = a\, \vec{L}\cdot\vec{S} - b\, (\vec{L} + 2\vec{S})\cdot\vec{B}$$

where a and b are positive constants, and $\langle H_0 \rangle = E_0$ for all the $n = 2$ levels of the Hydrogen atom. For this problem, always work in the basis $|\ell\ s\ m_\ell\ m_s\rangle$.

(a) Determine all the nonvanishing matrix elements of the form $\langle \ell\ s\ m'_\ell\ m'_s|V|\ell\ s\ m_\ell\ m_s\rangle$.

[10]

(b) When $\vec{B} = 0$, determine the energy levels and express the corresponding eigenstates in the basis $|\ell\ s\ m_\ell\ m_s\rangle$. (You may name them $|\alpha_1\rangle, |\alpha_2\rangle$, etc.) Among which of these pairs of states $|\alpha_i\rangle \leftrightarrow |\alpha_j\rangle$ will there be transitions due to the magnetic field ?

[10]

(c) Under what conditions can the effect of the magnetic field be treated as a perturbation ? Assuming this condition to be satisfied, when the magnetic field is turned on, which of the above states $|\alpha_i\rangle$ will have the lowest expectation value of energy ? Calculate this energy.

[10]

[You may use $L\cdot S = L_z S_z + (L_+ S_- + L_- S_+)/2$]

4. An electron in the ground state $|0\rangle$ of a one dimensional harmonic oscillator is placed in a region with uniform electric field. Find the first order correction to $|0\rangle$, and hence, determine the induced electric dipole moment.

[10]

Possibly useful relations:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \ , \ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

$$a|n\rangle = \sqrt{n}|n-1\rangle \ , \ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$\infty \quad \infty \quad \infty$