Dependence of the Hall coefficient on magnetic field

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In this work I shall show that if one considers the Hall effect in a material with two charge carriers and writes the transport equation in the relaxation time approximation then the Hall coefficient depends on the magnetic field in a very non-linear fashion. But the complicated formula reproduces the standard known expressions in the limits of high and low magnetic fields. Hence the dependence derived gives a testable detailed behaviour interpolating the limits.

For the particular case of the Germanium and $GaAs$, graphs have been plotted to see the symmetrical rise and fall of the coefficient around zero magnetic field and for $GaAs$ the fractional rise above the asymptotic value has been shown as a function of temperatures near room temperature. Hence one sees that for $GaAs$ the predicted behaviour can possibly be experimentally tested easily.

I. THE EQUATIONS OF MOTION

The initial set up is that of doing a Hall Effect measurement in a system which has two equally oppositely charged carriers moving through the material. The complicated motion they have because of collisions among themselves and with the rest of the material (in the cases of interest the background is assumed to be a static lattice) is assumed to be contained in the extra retarding terms in the equation of motion which has justification in being thought of as is known in literature as the "relaxation time approximation" of the Boltzmann transport equation. To start off speeds are assumed to be non-relativistic.

Later one would try to rationalize the limits of applicability of the following equations in the semiconductors of eventual interest.

Let the 2 carriers in the material be electrons and holes of equal and apposite charges $-e$ and $e > 0$. Let their masses be m_e and m_h and velocities be $\vec{v_e}$ and $\vec{v_h}$ and their relaxation times be τ_e and τ_h respectively. Then in the presence of a electric field $\vec{E}$ and a magnetic field $\vec{B}$ their equation of motion are,

$$m_e \left(\dot{\vec{v_e}} + \frac{\vec{v_e}}{\tau_e} \right) = -e \left(\vec{E} + \vec{v_e} \times \vec{B} \right)$$

$$m_h \left(\dot{\vec{v_h}} + \frac{\vec{v_h}}{\tau_h} \right) = e \left(\vec{E} + \vec{v_h} \times \vec{B} \right)$$

In the common experimental scenario the material on which the Hall effect will be done is in a thin rectangular strip and its plane can serve as a natural choice for the $x - y$ plane and the direction orthogonal to it is the z-direction along which the magnetic field is placed.

So let the components of the vectors be $\vec{E} = (E_x, E_y, E_z)$, $\vec{B} = (0, 0, B)$, $\vec{v_e} = (v_{e_x}, v_{e_y}, v_{e_z})$ and $\vec{v_h} = (v_{h_x}, v_{h_y}, v_{h_z})$. In steady state let $\dot{\vec{v_e}} = 0 = \dot{\vec{v_h}}$ and them the equations of motion will look like,

$$\vec{v_e} = -\frac{e\tau_e}{m_e} \left(\vec{E} + \vec{v_e} \times \vec{B} \right)$$

$$\vec{v_h} = \frac{e\tau_h}{m_h} \left(\vec{E} + \vec{v_h} \times \vec{B} \right)$$

Therefore the equations of motion in component form look like,

$$(v_{e_x}, v_{e_y}, v_{e_z}) = -\frac{e\tau_e}{m_e}((E_x, E_y, E_z) + (v_{e_y}B, -v_{e_x}B, 0))$$

$$(v_{h_x}, v_{h_y}, v_{h_z}) = \frac{e\tau_h}{m_h}((E_x, E_y, E_z) + ((v_{h_y}B, -v_{h_x}B, 0))$$

hence we have for each of the components,

$$v_{e_z} = -\frac{e\tau_e}{m_e}E_z, v_{h_z} = \frac{e\tau_h}{m_h}E_z$$

$$v_{e_x} = -\frac{e\tau_e}{m_e}\left[E_x + v_{e_y}B\right], v_{h_x} = \frac{e\tau_h}{m_h}\left[E_x + v_{h_y}B\right]$$

$$v_{e_y} = -\frac{e\tau_e}{m_e}\left[E_y - v_{e_x}B\right], v_{h_y} = \frac{e\tau_h}{m_h}\left[E_y - v_{h_x}B\right]$$

One defines the mobilities μ_e and μ_h for the electron and the hole respectively as $\mu_e = \frac{e\tau_e}{m_e}$ and $\mu_h = \frac{e\tau_h}{m_h}$ Hence solving the above set of simultaneous equations we have,

$$v_{e_y} = -\frac{\mu_e(E_y + (\mu_e B)E_x)}{1 + (\mu_e B)^2}$$

$$v_{e_x} = -\frac{\mu_e(E_x - (\mu_e B)E_y)}{1 + (\mu_e B)^2}$$

$$v_{h_y} = \frac{\mu_h(E_y - (\mu_h B)E_x)}{1 + (\mu_h B)^2}$$

$$v_{h_x} = \frac{\mu_h(E_x + (\mu_h B)E_y)}{1 + (\mu_h B)^2}$$

II. THE EQUATION OF THE CURRENTS

Let the current density vector be $\vec{j} = (j_x, j_y, j_z)$ and the concentration of holes be n_h and for the free electrons be n_e, then we have the general equations,

$$j_x = n_h e v_{h_x} - n_e e v_{e_x}$$

$$j_y = n_h e v_{h_y} - n_e e v_{e_y}$$

$$j_z = n_h e v_{h_z} - n_e e v_{e_z}$$

Substituting the expressions for the velocities from the last section one gets,

$$j_x = e\left[\frac{n_h\mu_h}{1 + (\mu_h B)^2} + \frac{n_e\mu_e}{1 + (\mu_e B)^2}\right]E_x + e\left[\frac{n_h\mu_h^2 B}{1 + (\mu_h B)^2} - \frac{n_e\mu_e^2 B}{1 + (\mu_e B)^2}\right]E_y$$

$$j_y = e\left[-\frac{n_h\mu_h^2 B}{1 + (\mu_h B)^2} + \frac{n_e\mu_e^2 B}{1 + (\mu_e B)^2}\right]E_x + e\left[\frac{n_h\mu_h}{1 + (\mu_h B)^2} + \frac{n_e\mu_e}{1 + (\mu_e B)^2}\right]E_y$$

$$j_z = e\left[n_h\mu_h + n_e\mu_e\right]E_z$$

Define $\xi_h = \frac{\mu_h B}{1+(\mu_h B)^2}$ and $\xi_e = \frac{\mu_e B}{1+(\mu_e B)^2}$ and then one can define the conductivity tensor σ as $\vec{j} = \sigma(\vec{E})$. Rearranging the above expressions for the components of $\vec{j}$ one has,

$$\begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix} = \begin{bmatrix} \frac{e(n_h\xi_h + n_e\xi_e)}{B} & e(n_h\mu_h\xi_h - n_e\mu_e\xi_e) & 0 \\ -e(n_h\mu_h\xi_h - n_e\mu_e\xi_e) & \frac{e(n_h\xi_h + n_e\xi_e)}{B} & 0 \\ 0 & 0 & e(n_h\mu_h + n_e\mu_e) \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

One notes that $\sigma(B) \neq \sigma(-B)$

III. THE FIELDS AT EQUILIBRIUM

In usual experimental situations $E_z = 0$ and the Hall-coefficient is measured when $j_y = 0$. This implies that one has,

$$E_x = \frac{(n_h\xi_h + n_e\xi_e)}{B(n_h\mu_h\xi_h - n_e\mu_e\xi_e)}E_y$$

Substituting this in the equation for j_x one has,

$$j_x = \left(\frac{e}{B^2}\right)\left[\frac{(n_h\xi_h + n_e\xi_e)^2 + B^2(n_h\mu_h\xi_h - n_e\mu_e\xi_e)^2}{(n_h\mu_h\xi_h - n_e\mu_e\xi_e)}\right]E_y$$

The **Hall-Coefficient** defined as $R_H = \frac{E_y}{j_x B}$ and that evaluates to,

$$R_H = \left(\frac{B}{e}\right)\left[\frac{(n_h\mu_h\xi_h - n_e\mu_e\xi_e)}{(n_h\xi_h + n_e\xi_e)^2 + B^2(n_h\mu_h\xi_h - n_e\mu_e\xi_e)^2}\right]$$

IV. HALL-COEFFICIENT

To make the dependence on the magnetic field explicit one can write it as,

$$R_H = \left(\frac{B}{e}\right)\left[\frac{\frac{n_h\mu_h^2 B}{1+(\mu_h B)^2} - \frac{n_e\mu_e^2 B}{1+(\mu_e B)^2}}{\left(\frac{n_h\mu_h B}{1+(\mu_h B)^2} + \frac{n_e\mu_e B}{1+(\mu_e B)^2}\right)^2 + B^2\left(\frac{n_h\mu_h^2 B}{1+(\mu_h B)^2} - \frac{n_e\mu_e^2 B}{1+(\mu_e B)^2}\right)^2}\right]$$

On the above expression one can take the two possible extreme limits to see that,

- For $B = 0$ one gets $R_H = \frac{n_h\mu_h^2 - n_e\mu_e^2}{e(n_h\mu_h + n_e\mu_e)^2}$ and hence in this limit recovers the equation of Problem 3 of Chapter 6 of Charles Kittel's "Introduction to Solid State Physics"
- For the limit of $B \longrightarrow \infty$ one sees that $R_H \longrightarrow \frac{1}{e(n_h - n_e)}$ and hence two dimensional magnetoresistance diverges linearly.
- The conceptual points that emerges are that,
  - The low magnetic field value of the Hall coefficient is a function of both the carrier's concentration and the mobility but the asymptotic value is not sensitive to the mobilities and it just depends on the difference of concentrations.
  - The Hall Coefficient shows a symmetrical peak around the zero magnetic field point.

Now one can try to look for materials where experimentally it is feasible to check the detailed non-linear behaviour of the Hall-Coefficient with magnetic field as predicted in this calculation. Here I show two extreme examples of Germanium and $GaAs$ where this effect can be seen to happen but the numerical values of the changes being orders of magnitude apart.

V. LIMITS OF VALIDITY

One of the obvious scenarios when the above equations shall fail to remain realistic is when the magnetic fields are so strong that they cause inter-band transitions. An estimate of the upper limit on the magnetic field coming from such considerations can be found by asking at what value of the magnetic field is the quantum of energy corresponding to the cyclotron frequency equal to the band gap of the semiconductor? This has slight subtleties as can be seen in Appendix J of the book by Ashcroft and Mermin. Finally the criteria comes to be $\frac{\hbar qB}{m} \ll \frac{E_g^2}{E_f}$ using standard symbols.

For say $GaAs$ one can use the value that $E_g = 1.42eV$ and $E_f = 4.78eV$ and the upper limit on the magnetic field comes out to be around $299T$.

Eventually it will be seen that the relevant effects are visible well below this upper limit for inter-band transitions and hence one need not worry about "magnetic breakthrough".

Further one notices that since here constant magnetic fields are considered by definition they will not show any variation over length scales of electron localization and hence one can justify using classical equations of motion.

VI. GERMANIUM

The following is the list of constants that were used to see the behaviour of the Hall-Coefficient for Germanium,

- $n_e = 10^{21}/cm^3$
- $n_h = 1.8 \times 10^{12}/m^3$
- $\mu_e = 0.45 m^2/Vs$
- $\mu_h = 0.35 m^2/Vs$

For this case the function has been plotted and the value of the magnitude of the Hall Coefficient at low magnetic fields tends to almost 0.006249999975694443 and asymptotically tends to almost 0.00625000001124.

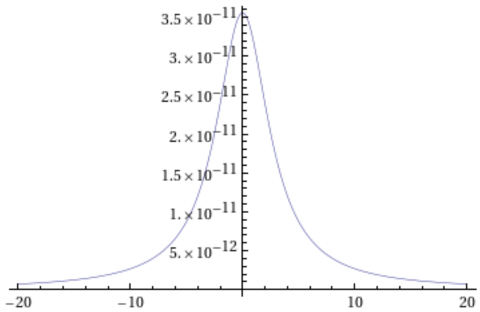

FIG. 1: The above graph shows the rise and fall of the Hall Coefficient (in S.I units) above the asymptotic magnitude of 0.00625000001124999 (in S.I units) as a function of magnetic field (in Tesla)

The Hall coefficient shows a symmetrical peak about the zero magnetic field point. And the fractional height of the peak above the asymptotic value is of the order of 10^{-9} and the rise above the asymptotic value is of the order of 10^{-11} .

The following is the order by order behaviour of the magnitude of the Hall Coefficient as a function of the magnetic field,

- Expanding about $B = 0$, for low magnetic fields it goes as

$$R_H(B) = 0.006249999975694443 - 4.35556 \times 10^{-12} B^2 + 5.33556 \times 10^{-13} B^4 + O(B^6)$$

- Expanding about $\frac{1}{B} = 0$, for high magnetic fields it goes as

$$R_H(\frac{1}{B}) = 0.00625000001124 + 2.90249 \times 10^{-10} \frac{1}{B^2} + 2.36938 \times 10^{-9} \frac{1}{B^4} + O(\frac{1}{B^6})$$

One sees that the magnitude of the maximum rise of Hall Coefficient if defined as $rise = R_H(B \to 0) - R_H(B \to \infty) = 3.55556 \times 10^{-11}$ and the fractional rise above the asymptotic value if defined as $\frac{rise}{|R_H(B \to \infty)|} = 5.68889 \times 10^{-9}$

The dependence of E_y as a function of the magnetic field can be seen to be predominantly linear as in the graph below as also seen from expanding E_y in a power series in B about $B = 0$ which gives,

$$E_y(B) = -208.305B + 4.14758 \times 10^{-8} B^3 - 5.08079 \times 10^{-9} B^5 + O(B^6)$$

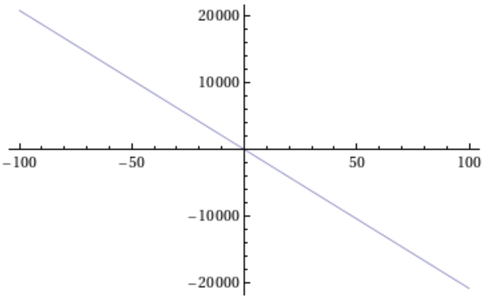

FIG. 2: The above graph shows E_y in Volts in the y-axis as a function of B in Teslas in the x-axis.

The dependence of j_x as a function of the magnetic field in the graph below can be seen to reflect the eventual rise and fall behaviour.

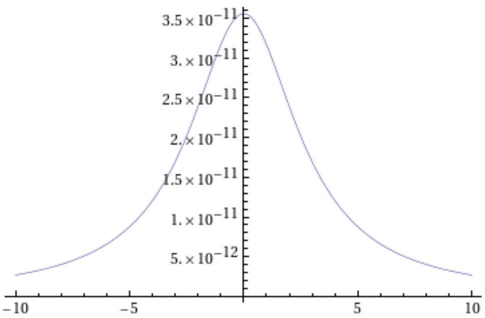

FIG. 3: The above graph shows rise and fall of j_x in A/m^2 above a base value of $33328.8 A/m^2$ in the y-axis as a function of B in Teslas in the x-axis.

VII. GALLIUM ARSENIDE

The fractional change of the Hall Coefficient for the Germanium is a bit too low for it to be experimentally detectable easily. A better candidate for seeing this change is $GaAs$.

One takes the following values for the parameters of $GaAs$,

- The mobility of the electrons shows a temperature dependence like $\mu_e(T) = \frac{0.94\times300}{T} m^2/(Vs)$ near $T = 300K$ and the mobility of the holes is like $\mu_h = 0.04m^2/(Vs)$
- One can choose to work at the Gamma-Valley at which $GaAs$ has a direct band gap of $E_g = 1.42eV$ and choose the maxima of the valence band as the zero of energy i.e $E_v = 0$ and hence one has for the the energy at the conduction band minima to be $E_c = E_g$.
- Let m be the mass of the electron and then the mass of the electronic excitation at the Gamma Valley is known to be $m_e = 0.063m$ and the typical mass of the hole excitation is known to be $m_h = 0.5m$.

To see the dependence of the concentration of carriers as a function of temperature let us take the simple approximate dependencies derived in the Chapter 28 of the book "Solid State Physics" by Ashcroft and Mermin as quoted below,

k_B is the Boltzmann's constant whose value is being taken as $8.617 \times 10^{-5}\ eV/K$ and μ is the chemical potential which is being measured in eV like E_c, E_g and E_v.

- Let n_e be the concentration of the electrons and that is given as,

$$n_e(T) = 2.5\left(\frac{m_e}{m}\right)^{1.5}\left(\frac{T}{300K}\right)^{1.5} \exp^{\frac{-(E_c-\mu)}{k_BT}} \times 10^{25}\ /m^3$$

- Let n_h be the concentration of the holes and that is given as,

$$n_h(T) = 2.5\left(\frac{m_h}{m}\right)^{1.5}\left(\frac{T}{300K}\right)^{1.5} \exp^{\frac{-(\mu-E_v)}{k_BT}} \times 10^{25}\ /m^3$$

- At similar levels of approximation one uses the following expression for the dependence of the chemical potential with temperature,

$$\mu(T) = E_v + \frac{1}{2}E_g + \frac{3}{4}k_BTln\left(\frac{m_h}{m_e}\right)$$

The fractional rise above the asymptotic value of the Hall Coefficient as a function of temperature can be defined as $\frac{R_H(B\to0)-R_H(B\to\infty)}{|R_H(B\to\infty)|}$ where now there are temperature dependencies as,

- $R_H(B \to 0) = \frac{n_h(T)\mu_h^2 - n_e(T)\mu_e(T)^2}{e(n_h(T)\mu_h + n_e(T)\mu_e(T))^2}$
- $R_H(B \to \infty) = \frac{1}{e(n_h(T)-n_e(T))}$

Using the above temperature dependencies the variation of the fractional rise as a function of temperature looks like,

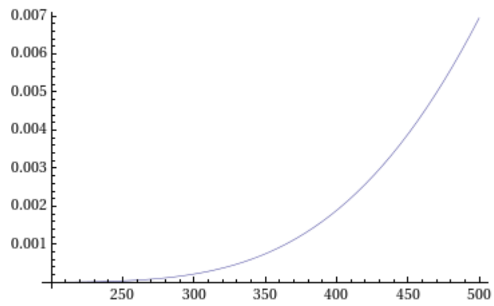

FIG. 4: The above graph shows on the y-axis the fractional rise of the Hall Coefficient above the asymptotic value for $GaAs$ versus temperature in Kelvin from $200K$ and $500K$

For $GaAs$ the variation of the Hall Coefficient with magnetic field above the asymptotic value at $300K$ is about 10^{10} times larger than that for Germanium and the peak is also much broadened out. One sees in the graph below the similar bell-shape of the curve as for Germanium only when one scans from $-100T$ to $100T$ unlike appearance of the same curve for Germanium on scanning only from $-20T$ to $20T$.

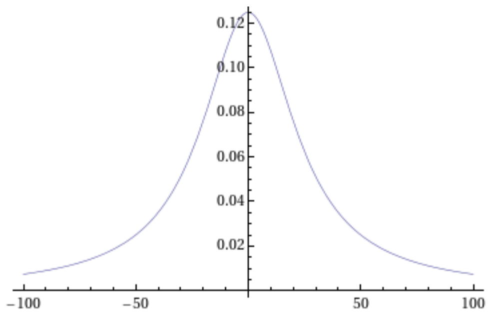

FIG. 5: Variation of the Hall-Coefficient in the y-axis for $GaAs$ at $300K$ above its asymptotic value of $R_H(B \to \infty) = -570.493$(in S.I units) as a function of magnetic field in Teslas in the x-axis

VIII. CONCLUSIONS

We see that simple relaxation-time approximation of the transport equation for holes and electrons for semiconductors can give a non-trivial dependence of the Hall Coefficient on the magnetic field and this symmetrical rise and fall of the coefficient around zero magnetic field is within easily measurable regimes for materials like $GaAs$ where fractional changes are of the order of 10^{-3} although it can be a very small effect for materials like Germanium where the fractional changes are of the order of 10^{-9} . The complicated formula as expected, for high magnetic field asymptotically goes to a value independent of the mobilities of the carriers whereas the low magnetic field limiting value of the Hall Coefficient is dependent on the mobilities and the concentrations of the carriers.

IX. ACKNOWLEDGEMENTS

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[1] "Solid State Physics" by Ashcroft and Mermin
[2] "Introduction to Solid State Physics" by Charles Kittel