Notes on "large N" $\mathcal{N} = 2$ supersymmetric Chern-Simons-matter theory

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INTRODUCTION

This is a report on some of the calculations that were done during the second half of my one year MSc. internship since August 2010. During the first half of the internship, I was studying supersymmetry [14, 15] and superconformality [11–13]. In the following paragraphs I shall briefly review the research developments which led to my explorations during the later half of the internship.

In a beautiful paper [4] John Schwartz had pointed out the possibility of being able to supersymmetrize the Chern-Simons theory coupled to matter. Interestingly with even low amounts of supersymmety this theory already gives a natural example of the phenomenon of superconformality. Otherwise in $3 + 1$ dimensions the ubiquitous scenario is of conformality being broken by quantum corrections though with high amounts of supersymmetry like in $\mathcal{N} = 4$ SYM that can be saved. In recent times there has been an intense interest in looking at superconformal field theories, like most notably [3] (the so called "ABJM theory" by Ofer Aharony, Oren Bergman, Daniel Louis Jafferis and Juan Maldacena) These theories because of their high degree of symmetry give an increased analytic control for calculating quantities which are very likely to be deformation invariant and hence can provide a way of checking conjectured strong/weak coupling dualities like the "AdS/CFT" conjecture by Maldacena as reviewed here [19]. In the seminal paper [2] Davide Gaiotto and Xi Yin among other things had brought to light the exciting properties of a particular superconformal field theory namely the $\mathcal{N} = 2$ super-Chern-Simons-matter theory in $2 + 1$ dimensions and that will be of central interest in this article. One of the interesting things that Davide Gaiotto and Xi Yin had argued in that paper is that the matter coupling in their Lagrangian flows under renormalization such that there is a conformal fixed point and there the supersymmetry enhances to $\mathcal{N} = 3$!

In parallel to the above developments in another recent fascinating paper [5], the authors Daniel Green, Zohar Komargodski, Nathan Seiberg, Yuji Tachikawa and Brian Wecht gave a geometric description for describing the so called "conformal manifold" of $\mathcal{N} = 1$ supersymmetric field theory in $3 + 1$ dimensions. This conformal manifold arises as the space of all exactly/quantum theoretically marginal deformations of a supersymmetric field theory around its conformal fixed point. This work builds on the path breaking foundational work on the subject by Rob Leigh and Matthew Strassler [7].

By the geometric prescription of [5], for the particular case of my interest - $\mathcal{N} = 2$ super-Chern-Simons theory coupled to matter, the conformal manifold should be described as the quotient of the space of marginal couplings (here $\mathbb{C}^5$) by the complexified global Lie symmetry group (here $SU(2)^{\mathbb{C}} = SL(2, \mathbb{C})$). During this project investigation was done to understand this quotient space and some progress was made using Geometric Invariant Theory (GIT) [17] but this part of the efforts are not being reported here.

There are strong reasons to believe that the spectrum of BPS states is also independent of the 'tHooft coupling in the 'tHooft limit and hence being able to calculate that in various supersymmetric theories has been the focus of many recent enlightening papers like [22–24] by Christian Romelsberger, F.A.Dolan, Bo Feng, Amihay Hanany and Yang-Hui He. Many of the combinatorial techniques which permeate these papers and the related insights about the simplifications that happen in the "large N" limit were pointed out in an extremely insightful paper by Polyakov [20].

If one chooses a supercharge then in superconformal theories it acts as an endomorphism on the space of states at a fixed value of "energy" (scaling dimension + $z-$component of spin). It is believed that for a chosen supercharge (say Q) and an energy level, the equivalence classes, $Ker(Q)/Im(Q)$, of the above action, calculated in the classical theory are in bijection to the BPS states at that energy level. Similar motivational reasons, as behind the works cited above, lead one to believe that this "Q-cohomology" is also invariant under deformations and hence calculating that paves another way to test various dualities.

In this article I shall give an in-principle method of being able to write down the "exact" number of states at a fixed value of "energy". An explicit expression will be given to achieve it and to evaluate it an approximation technique will be explained though I don't know of a closed form solution. One way to test the quite simple approximation technique will be to see that the Hagedorn transition temperature that it predicts agrees very closely with the exact results for this temperature (obtained by much more sophisticated techniques) as obtained in this paper [21] by Ofer Aharony, Joseph Marsano, Shiraz Minwalla, Kyriakos Papadodimas and Mark Van Raamsdonk. I shall also report some data that was calculated for the first few "Q-cohomologies".

This article is strongly influenced by this recent paper [1] by Shiraz Minwalla, Prithvi Narayan, Tarun Sharma, V. Umesh and Xi Yin. I shall be largely using the notations as in the above paper and some of the related observations from that paper are summarized in the conclusion of this article.

THE FIELDS AND THEIR QUANTUM NUMBERS

In a superconformal theory one can choose as the simultaneously specifiable quantum numbers the following attributes of the fields/operators, namely scaling dimension (Δ), the "z" component of the spin (j) and the $R-$charge (h). For every operator one can write the $3-$tuple of the above in the order, (Δ, j, h). I shall be writing all my supersymmetry equations on-shell and hence I have only a scalar field ϕ and a fermionic field ψ. We work at a chosen solution such that at every spacetime point a particular component of the ψ is chosen and that shall be denoted as χ. Similarly the letter Q and D shall denote a particular component of the supersymmetry operator and the superderivative with the specific (Δ, j, h) as stated below.

operator	(Δ, j, h)
ϕ	$(\frac{1}{2}, 0, \frac{1}{2})$
χ	$(1, \frac{1}{2}, \frac{1}{2})$
Q	$(\frac{1}{2}, -\frac{1}{2}, 1)$
D	$(1, 1, 0)$

One notes that these "charges" are additive over taking product of operators since they transform as product of exponentials. Hence if the charges for an operator O are (Δ, j, h) then that for the $Q(O)$ will be $(\Delta + \frac{1}{2}, j - \frac{1}{2}, h + 1)$. Also note that the sum $\Delta + j$ is conserved by the action of supersymmetry and hence that will be the parameter by which the Witten Index will be weighted and also the parameter using which I will organize the list of all possible operators.

Though the particular sequence in which the operators occur in a composite operator is important (up to cyclic permutation because of the overall trace over gauge index in the actual operators) for the sake of counting its quantum number it is not. Hence for the sake of counting quantum numbers the most general operator can be thought of as "$\phi^n \chi^m D^k$" with the quantum numbers $(\frac{n}{2} + m + k, \frac{m}{2} + k, \frac{n}{2} + \frac{m}{2})$. Then after the action of the supersymmetry operator the quantum numbers of $Q(\phi^n \chi^m D^k)$ are $(\frac{n}{2} + m + k + \frac{1}{2}, \frac{m}{2} + k - \frac{1}{2}, \frac{n}{2} + \frac{m}{2} + 1)$. Hence the action of Q preserves the value of $\Delta + j$ as $\frac{n}{2} + \frac{3m}{2} + 2k$. Coupled to the fact that Q is linear over "words" (states/operators) in the "letters" ϕ, χ and D it is consistent to talk of the matrix of Q at every value of $\Delta + j$.

If given an operator O_1 there exists another operator O_2 such that $O_1 = Q(O_2)$ the O_1 will be called $Q - exact$. If for an operator O it is true that $Q(O) = 0$ then O will be called $Q - closed$.

Quantum number constraint on being $Q - exact$

Let there exist whole numbers n, n', m, m', k, k' such that, $\phi^{n'} \chi^{m'} D^{k'} = Q(\phi^n \chi^m D^k)$ then by matching the values of Δ, j and h on either side one has the following equations,

$$m' + k' + \frac{n'}{2} = m + k + \frac{n}{2} + \frac{1}{2}$$

$$\frac{m'}{2} + k' = \frac{m}{2} + k - \frac{1}{2}$$

$$\frac{n'}{2} + \frac{m'}{2} = \frac{n}{2} + \frac{m}{2} + 1$$

One can "solve" the above to get,

$$m' = (m + n + 2) - n'$$

$$k' = (k - \frac{n}{2} - \frac{3}{2}) + \frac{n'}{2}$$

since m' and k' need to be ≥ 0 the above effectively says that,

$$0 \leq n' \leq min(m + n + 2, n + 3 - k)$$

ACTION OF THE SUPERSYMMETRY OPERATOR

For the theory at hand the action of the supersymmetry operator on the fields is given as,

- $Q\phi = [Q, \phi] = 0$
- $Q\chi = \{Q, \chi\} = 0$
- $Q(DO) = [[\phi, \chi], O\} + D(QO)$
  where,
  - O is any arbitrary operator made up of only ϕ and χ.
  - The notation $[, \}$ means that it will be a commutator or an anti-commutator depending on whether O is bosonic or fermionic respectively. O will be bosonic or fermionic depending on whether it has even or odd number of χs in it.
- Also if A and B are two operators made up of ϕ, χ and D only and $f(A)$ be the number of fermions (χs) in A then D acts as a normal derivative on the product AB and Q acts like a graded derivative as follows,

$$D(AB) = (DA)B + A(DB)$$

$$Q(AB) = (QA)B + (-1)^{f(A)}A(QB)$$

- One postulates that if O is any arbitrary state/operator and $D(O) = 0$ then $O = 0$. This can be rationalized by saying that any constant field can always be gauged away and that is consistent since D is actually the gauge covariant derivative.

Some immediate observations follow from the above,

- In the "large N" limit, there are no trace relations and hence apart from cyclic permutation of fields there is no other symmetry among the states. This greatly simplifies the analysis.
- Any state is heuristically a trace over a string ϕ, χ and D matrices. This means that if the "letters" are cyclically permuted then because of the fermionic nature of χ the resultant expression can differ from the original by an overall minus sign. This is implemented by multiplying the state by a -1 every time a χ crosses another χ (or Q). Like $Tr[\phi\chi D(\phi)\chi] = -Tr[\chi\phi\chi D(\phi)]$. This means that the result of the action of Q on various members of an equivalence class of strings of letters up to cyclic permutation can only differ by an overall sign. This will be tantamount to flipping all the signs of one or more columns in the matrix for Q at any value of $\Delta + j$. But doing this does not change the rank or the dimension of the kernel of the matrix - the only two properties of the matrix that will be of relevance here. Hence it is consistent to live with the sign ambiguity resulting from arbitrarily picking the representative from each equivalence class under cyclic permutation of the "words" in the "letters".

- If O is any arbitrary state/operator then one can show that $Q(DO) = 0$ iff $Q(O) = 0$. This implies that at any value of $\Delta + j$ one can restrict to only those states which are not total derivatives since any $Q-$closed total derivative state is necessarily a "descendant" of some $Q-$closed state at $\Delta + j - 2$ and hence in some sense it is not a "new" state. The above can be proven by separately considering the cases of O being bosonic or fermionic and to prove that $Q(DO) = 0$ implies $Q(O) = 0$ one will crucially need the property that if $D(O) = 0$ then $O = 0$.

Steps towards understanding the kernel of Q

One dreams of being able to classify all possible linearly independent states or words which are annihilated by Q. This as of now has not been possible and efforts are on to try to write down in some sense a list of all "words" in the kernel of Q. But a priori it seems that for any non-negative integer n states of the following type are annihilated by Q like ϕ^n, $\phi^n D(\phi)$, $D^n\phi$, $\phi^n\chi^2$ and $\phi^n\chi\phi\chi$.

That the above states map to 0 under the action of Q is easy to check by explicit calculation and at least till the first 10 values of $\Delta + j$ no other word of any other form has been found which is mapped to 0. Though one should note that these are not all states that are in the kernel since there are bunches of states none of which annihilate under Q but some linear combination of them does. These will definitely contribute to the kernel but for a classification program it doesn't seem wise to hunt for such states since they are not be linearly independent.

PROVING THAT THE NILPOTENCY INDEX OF Q IS 2

Any arbitrary state in the theory is a word in the alphabets ϕ, χ and D. By gauge invariance the words are defined under an overall gauge trace and hence are physically identified up to cyclic permutations. The words can obviously be arbitrarily long.

So one wants to prove that the operator Q^2 as defined above annihilates any such word.

One can think of the words being created in a recursive way as follows. Imagine a countably infinite number of separate trees being defined with a ϕ, χ, $D^n\phi$ and $D^m\chi$ as their roots. (for all $n, m \in \mathbb{Z}^+$) And the next level of nodes in the trees will be obtained by appending a ϕ or χ or $D^n\phi$ or $D^m\chi$ to the right of the words in the last level. One can easily see that constructed this way every possible word will occur somewhere in one of the trees. All the trees will be mathematically disjoint but by cyclicity the words on many of the nodes will physically be the same states of the theory.

One can enhance the scope of the proof by generalizing the defined action of Q by assuming that the alleged supersymmetry transformations come from a theory which has a $Tr(\phi^4)$ superpotential. Hence the action of Q on χ is given as $Q\chi = \phi^3$. Then the action of Q on the basic derivative states would be, $Q(D\phi) = 2\phi\chi\phi - \chi\phi^2 - \phi^2\chi$ and $Q(D\chi) = \phi\chi^2 - \chi^2\phi + D(\phi^3)$. By another iteration of the action it follows that $Q^2(D\phi) = Q^2(D\chi) = Q^2(\phi) = Q^2(\chi) = 0$. This shall be used as a base case for the induction argument to prove that $Q^2(\textit{any word}) = 0$

Assuming that $Q^2(D^{n-1}\phi) = 0$ one can show as follows that $Q^2(D^n\phi) = 0$

$$Q(D^n\phi) = Q(D(D^{n-1}\phi)) = [[\phi, \chi], D^{n-1}\phi] + D(Q(D^{n-1}\phi))$$

So,

$$Q^2(D^n\phi) = Q([[\phi, \chi], D^{n-1}\phi]) + \{[\phi, \chi], Q(D^{n-1}\phi)\} + D(Q^2(D^{n-1}\phi))$$

$$= Q((\phi\chi - \chi\phi)D^{n-1}\phi - D^{n-1}\phi(\phi\chi - \chi\phi)) + (\phi\chi - \chi\phi)Q(D^{n-1}\phi) + Q(D^{n-1}\phi)(\phi\chi - \chi\phi)$$

$= \phi^4 D^{n-1}\phi - \phi\chi Q(D^{n-1}\phi) - \phi^4 D^{n-1}\phi + \chi\phi Q(D^{n-1}\phi) - Q(D^{n-1}\phi)\phi\chi - (D^{n-1}\phi)\phi^4 + Q(D^{n-1}\phi)\chi\phi + (D^{n-1}\phi)\phi^4 + (\phi\chi - \chi\phi)Q(D^{n-1}\phi) + Q(D^{n-1}\phi)(\phi\chi - \chi\phi)$

$= 0$

Similarly assuming that $Q^2(D^{n-1}\chi) = 0$ one can show that $Q^2(D^n\chi) = 0$

$$Q(D^n\chi) = Q(D(D^{n-1}\chi)) = \{[\phi,\chi], D^{n-1}\chi\} + D(Q(D^{n-1}\chi))$$

So,

$$Q^2(D^n\chi) = Q((\phi\chi - \chi\phi)D^{n-1}\chi + D^{n-1}\chi(\phi\chi - \chi\phi)) + (\phi\chi - \chi\phi)Q(D^{n-1}\chi) - Q(D^{n-1}\chi)(\phi\chi - \chi\phi)$$

$= \phi^4 D^{n-1}\chi - \phi\chi Q(D^{n-1}\chi) - \phi^4 D^{n-1}\chi + \chi\phi Q(D^{n-1}\chi) + Q(D^{n-1}\chi)\phi\chi - (D^{n-1}\chi)\phi^4 - Q(D^{n-1}\chi)\chi\phi + (D^{n-1}\chi)\phi^4 + (\phi\chi - \chi\phi)Q(D^{n-1}\chi) - Q(D^{n-1}\chi)(\phi\chi - \chi\phi)$

$= 0$

Now that one has shown that Q^2 annihilates all the roots of the trees, let A be a generic word at any level of the tree and let $f(A)$ be the number of fermions (χs) in A. Hence a factor of $(-1)^{f(A)}$ will be picked up every time Q passes through it. Inductively assuming that $Q^2(A) = 0$ one wants to show that so is true on all possible words in the next level i.e $Q^2(A\phi) = Q^2(A\chi) = Q^2(AD^n\phi) = Q^2(AD^m\chi) = 0$. One can show each of them as follows,

$Q^2(A\phi) = (Q^2(A))\phi = 0$

$Q^2(A\chi) = Q((QA)\chi + A\phi^3(-1)^{f(A)}) = (Q^2(A))\chi + (-1)^{f(A)+1}(QA)(Q\chi) + (QA)\phi^3(-1)^{f(A)} = 0$

$Q^2(AD^n\phi) = Q(Q(A)D^n\phi + AQ(D^n\phi)(-1)^{f(A)}) = (Q^2(A))D^n\phi + (QA)Q(D^n\phi)(-1)^{f(A)+1} + (QA)Q(D^n\phi)(-1)^{f(A)} + AQ^2(D^n\phi)(-1)^{2f(A)} = 0$

$Q^2(AD^n\chi) = Q((QA)D^n\chi + AQ(D^n\chi)(-1)^{f(A)}) = (Q^2(A))D^n\chi + (QA)Q(D^n\chi)(-1)^{f(A)+1} + (QA)Q(D^n\chi)(-1)^{f(A)} + AQ^2(D^n\chi)(-1)^{2f(A)} = 0$

Hence one has shown that Q^2 annihilates every state.

LISTING OPERATORS AT EACH VALUE OF $\Delta + j$

As examples I shall be listing "all" the operators at some values of $\Delta + j$ and specifying which of them is $Q-closed$ since these are the one's that specify the $Q-cohomology$ classes.

- All operators are understood to have an overall gauge trace which will not be made explicit. Like $\phi^2 D\chi$ would mean implicitly $Tr[\phi^2 D\chi]$.
- Given the above all equalities are understood to hold up to cyclic permutations. That is I would write $Q(\phi^2\chi D\phi) = \phi^2\chi[[\phi,\chi],\phi] = \phi^2\chi\phi\chi\phi - \phi^2\chi^2\phi^2 - \phi^2\chi\phi^2\chi + \phi^2\chi\phi\chi\phi = -\phi^4\chi^2 - \phi^2\chi\phi^2\chi + 2\phi^3\chi\phi\chi$. In the above the last equality is implicitly gotten by cyclic permutation under the trace.
- As was pointed out by Polyakov in the paper mentioned in the introduction, the "large N" limit plays in here by removing trace relations between various powers of the operators. This simplifies the calculations by a lot.

The generic operator will be gotten by various permutations (up to cyclic permutations) of all the strings of the form $\phi^m\chi^m D^k$ such that it has the $\Delta + j$ being listed then i.e they satisfy, $\frac{n}{2} + \frac{3m}{2} + 2k = \Delta + j$

- All the operators gotten in this way will not be linearly independent and hence the linearly independent and the dependent ones will be listed separately. Like at $\Delta + j = 4$ one would get among others $D(\phi\chi)$, $\chi D\phi$ and $\phi D\chi$. The first of these will appear separately in the third column. Of course arbitrary linear combinations will not get listed because they can't in general be combined into one operator by using Leibnitz property of the D.

- Hence by the above in the columns for linearly independent operators never will occur a term where D is acting on an operator which contains more than one ϕ or χ.

- In this listing if two operators are produced such that one when expanded by using the Leibnitz property of the D gives some scalar multiple of the other then only one representative form of the term will be listed. For ease of counting purposes I will always list the term where D is acting on only one operator. Like at $\Delta + j = 4$ one would get among others three seemingly different operators, $D(\phi^3)$, $\phi D(\phi^2)$, $\phi^2 D(\phi)$. But one sees that $\frac{1}{3}D(\phi^2) = \frac{1}{2}\phi D(\phi^2) = \phi^2 D(\phi)$. Hence by the chosen convention only $\phi^2 D(\phi)$ will be listed in the list of linearly independent operators.

- When calculating the Witten Index one separately counts the contribution of the operators with an overall D (like $D(\chi\phi^3)$ at $\Delta + j = 5$). Operators with only D's will also be counted in this category.

- At least till $\Delta + j = 5$ it seems that one always gets a basis of operators none of which are $Q - exact$ and hence in this list I have not yet included a separate column of $Q - exact$ operators. Of course at this point I don't understand why this should happen or whether it continues to happen for higher values of $\Delta + j$.

$\Delta + j$	linearly independent $Q - closed$	linearly independent not $Q - closed$	linearly dependent not $Q - exact$
$\frac{1}{2}$	ϕ		
1	ϕ^2		
$\frac{3}{2}$	ϕ^3, χ		
2	$\phi\chi$, ϕ^4, (D ?)		
$\frac{5}{2}$	$\phi^2\chi$, ϕ^5, $D\phi$		
3	χ^2, $\phi^3\chi$, ϕ^6, $\phi D(\phi) = \frac{1}{2}D(\phi^2)$		
$\frac{7}{2}$	$\phi\chi^2$, $\phi^4\chi$, ϕ^7, $D(\chi)$, $\phi^2 D(\phi)$		
4	ϕ^8, $\phi^5\chi$, $\phi^2\chi^2$, $\chi\phi\chi\phi$, $\phi^3 D(\phi)$, $\phi D(\chi)$	$\chi D(\phi)$	$D(\phi\chi)$, D^2 (?)
$\frac{9}{2}$	ϕ^9, $\phi^6\chi$, $\phi^3\chi^2$, $\phi\chi\phi\chi\phi$, χ^3, $\phi^4 D(\phi)$, $\phi^2 D(\chi)$, $D^2\phi$	$\phi\chi D(\phi)$, $\chi\phi D(\phi)$	$\chi D(\phi^2)$, $\phi D(\chi\phi)$, $\phi D(\phi\chi)$, $D(\phi^2\chi)$, $D(\chi\phi^2)$
5	ϕ^{10}, $\phi^7\chi$, $\phi^4\chi^2$, $\phi^3\chi\phi\chi$, $\phi^2\chi\phi^2\chi$, $\phi\chi^3$, $\phi^5 D(\phi)$, $\chi D(\chi)$, $\phi^3 D(\chi)$	$\phi^2\chi D(\phi)$, $\chi\phi^2 D(\phi)$, $\phi\chi\phi D(\phi)$, $(D\phi)^2$, $\phi D^2(\phi)$(?)	$\phi^2 D(\phi\chi)$, $\phi^2 D(\chi\phi)$, $\phi D(\phi^2\chi)$, $\phi D(\phi\chi\phi)$, $\phi D(\chi\phi^2)$, $D(\chi\phi^3)$, $D(\phi^3\chi)$, $D(\phi\chi\phi^2)$, $D(\phi^2\chi\phi)$

- As of now I don't have an understanding of how many linearly independent terms (sum of the number of terms in the second and the third columns of the above table) there should be at every value of $\Delta + j$. Or in other words it is not clear as to how the dimension of the space of states changes with $\Delta + j$.
- As of now it is also not clear as to how does the number of equivalence classes of $Q - closed$ mod $Q - exact$ operators change with $\Delta + j$.

Ignoring the fermionic nature of χ

In the above list of states one can see that $\phi\chi\phi\chi\phi$ and $\phi^3\chi^2$ have been written down whereas these should be actually 0 by the fermionic nature of χ. In the counting of number of linearly independent states at ever $\Delta + j$ one can possibly account for this detail but the point is that eventually I would at best be able to make statements about the size of the vector space dimension of $\Delta + j$ in the large $\Delta + j$ limit. This vector space dimension is expected to show an exponential growth and the claim is that this asymptotic behaviour will remain unaffected by the inclusion of these small number of extra states which are actually 0.

STEPS TOWARDS COUNTING THE NUMBER OF OPERATORS AT EACH $\Delta + j$

A basic step towards getting an understanding of the $Q - cohomology$ is to try to understand the growth of number of states with increasing $\Delta + j$. (i.e the sum of the number of terms in the second and the third column of the above table) This itself turns out to be a challenging problem and only asymptotics seems feasible.

The three key observations that help are,

- At every $\Delta + j$ the basis consists of only such terms where there are either no D's or the D acts on just one field ϕ or χ.
- One notes that if a term consists of n ϕs, m χs and k Ds then its $\Delta + j$ is given by the equation, $n + 3m + 4k = 2(\Delta + j)$. Hence at each $\Delta + j$ the terms can be classified by the number of operators it has of each type i.e by the positive integral solutions of the above Diophantine equation.
- Given any one such term corresponding to a fixed positive integral solution of the above Diophantine equation, all the cyclic permutations of a string of letters correspond to the same operator since everything is under a trace. More importantly this is true even with a D. For example one can look at the term $\phi D(\chi)$ and then look at all the cyclic permutations of this string of letters, "$\phi D\chi$" and the other "terms" it will produce are "$\chi\phi D$" and "$D\chi\phi$". The crucial thing to observe is that all these permutations produce terms which are equivalent like $D(\chi)\phi$ or which are not-so-meaningful like "$\chi\phi D$". Hence it suffices to count permutations of strings of letters with fixed number of ϕs, χs and Ds up to cyclic permutation. (This could be reminiscent of what is called the "inclusion-exclusion" technique in combinatorics.)

As an example for a fixed $\Delta + j$ let me give the detailed table of how the above classification and the count looks like. Like for $\Delta + j = \frac{9}{2}$ the terms are classified by the positive integral solution of the Diophantine equation, $n + 3m + 4k = 9$ (the variables mean as defined above). Below is given the list of the linearly independent operators which are produced at each solution of this equation.

Note that all the operators occurring in the second column of this table are precisely the same operators which occurred in the second and the third column of the above table in the row corresponding to $\Delta + j = \frac{9}{2}$. Hence there are two natural ways of classifying the linearly independent operators at each $\Delta + j$ either by whether they are $Q - closed$ or not or by the solution of the Diophantine equation to which they correspond.

(n, m, k) tuple solving $n + 3m + 4k = 2(\Delta + j = \frac{9}{2})$	Linearly independent operators corresponding to this solution
$(9, 0, 0)$	ϕ^9
$(6, 1, 0)$	$\phi^6\chi$
$(3, 2, 0)$	$\phi^3\chi^2$, $\phi\chi\phi\chi\phi$
$(0, 3, 0)$	χ^3
$(5, 0, 1)$	$\phi^4 D(\phi)$
$(2, 1, 1)$	$\phi^2 D(\chi)$, $\phi\chi D(\phi)$, $\chi\phi D(\phi)$
$(1, 0, 2)$	$D^2(\phi)$

In practice firstly tables of the above kind were made for each $\Delta + j$ and then they were combined into a single table of the first kind. For ease of presentation it is here being reported in the reverse order.

From the above arguments it is clear that to make a count of the number of linearly independent operators at a given value of $\Delta + j$ one has to solve the following two questions,

- Given a 3-tuple of non-negative integers (n, m, k) one needs to count the number of "words" up to cyclic permutation that can be made using n ϕs, m χs and k Ds.
- For the given value of $\Delta + j$ one needs to sum over the above count for every solution tuple (n, m, k) of the Diophantine equation, $n + 3m + 4k = 2(\Delta + j)$.

The first question can be solved using the standard technique of Polya theory and for the second only asymptotic estimates can be made.

Counting the number of solutions to arbitrary linear Diophantine equations

It would have been desirable if there was a way to exactly be able to write down the number of (or even parametrize!) the positive integral solutions of the equation, $n + 3m + 4k = 2(\Delta + j)$. But the best one seems to be able to do is what is called the Schur's Theorem.

That if $a_1, a_2, ..., a_n$ are mutually coprime positive integers then the number of positive integral solutions of $a_1x_1 + a_2x_2 + ... + a_nx_n = N$ grows asymptotically as,

$$\frac{N^{n-1}}{(a_1a_2a_3...a_n)(n-1)!}$$

Hence in this case one can say that the number of solutions to the equation, $n + 3m + 4k = 2(\Delta + j)$ grows as,

$$\frac{(\Delta + j)^2}{6}$$

Counting the number of "words" up to cyclic permutation using n ϕs, m χs and k Ds

Assign a "weight" of X to the field ϕ, Y to the field χ and Z to D and then Polya's theory tells us that the number of words up to cyclic permutation that can be made using n ϕs, m χs and k Ds is precisely the coefficient of the term $X^nY^mZ^k$ in the sum,

$$\frac{1}{n+m+k}\Sigma_{d|n+m+k}\phi(d)[X^d + Y^d + Z^d]^{\frac{n+m+k}{d}}$$

(where ϕ is the Euler totient function)
By using the binomial theorem twice it can be seen that the required coefficient and hence the number of words up to cyclic permutation using n ϕs, m χs and k Ds is =,

$$\frac{1}{n+m+k}\Sigma_{d|n+m+k \text{ and } d|m+k \text{ and } d|m}\phi(d)^{\frac{n+m+k}{d}}C_{\frac{m+k}{d}} {}^{\frac{m+k}{d}}C_{\frac{m}{d}}$$

As an example one can evaluate the above expression for $n = 2$, $m = 1$ and $k = 1$ and one would get the answer as 3 and one sees that there are precisely 3 linearly independent operators corresponding to $(2, 1, 1)$ in the table above. Similarly for $n = 3$, $m = 2$ and $k = 0$ the above expression evaluates to 2 and there are just 2 linearly independent operators in the column corresponding to $(3, 2, 0)$. Similarly for $(0, 3, 0)$ the correct answer of 1 is gotten.

Hence the calculation that needs to be done to count the dimension of the vector space of operators at a given $\Delta + j$ is to evaluate the following expression,

$$\sum_{(n,m,k)\ s.t\ n+3m+4k=2(\Delta+j)} \frac{1}{n+m+k}\Sigma_{(d|n+m+k \text{ and } d|m+k \text{ and } d|m)}\ \phi(d)^{\frac{n+m+k}{d}}C_{\frac{m+k}{d}} {}^{\frac{m+k}{d}}C_{\frac{m}{d}}$$

and this can be rewritten as,

$$\sum_{(n,m,k)\ s.t\ n+3m+4k=2(\Delta+j)} \frac{1}{n+m+k}\Sigma_{(d|n,d|m,d|k)}\ \phi(d)\frac{\left(\frac{n+m+k}{d}\right)!}{\left(\frac{n}{d}\right)!\left(\frac{m}{d}\right)!\left(\frac{k}{d}\right)!}$$

Here on I shall explain a way of getting an asymptotic estimate of this expression. The central observation is that such multinomial coefficients maximize "near the middle" or around when they are balanced. Also common divisors, $d > 1$ will have their contributions to the sum hugely suppressed compared to the contribution coming from the term $d = 1$.

In the following way one can very well estimate what constitutes the "balanced solutions" of the Diophantine equations. One can always write the right hand side of the Diophantine equation as, $2(\Delta + j) = 8p + r$ for some $p \in \mathbb{Z}^+$ and $r \in \{0, 1, 2, ..., 7\}$. In that case there is always a countably infinite family of solutions of the form, $n = p + r$, $m = p$ and $k = p$. (Of course these are not all the solutions.) The convenient thing about this family is that the solutions here scale linearly with the right hand side $2(\Delta + j)$. Since the series one is looking at is a sum of positive terms the growth of these terms will always give a lower bound and these are expected to be the dominant contributors. It also turns out that these are enough to get some of the physics correct!

So the term corresponding to this solution family and $d = 1$ is,

$$\frac{1}{3p + r} \frac{(3p + r)!}{(p + r)!p!p!}$$

Since in the large $\Delta + j$ limit all of the above numbers will asymptotically grow similarly one can always use the Stirling approximation for them. Anyway the Stirling approximation (that $n! \sim n^n e^{-n}\sqrt{2\pi n}$) is not just an asymptotic estimate but a lower bound and as well as an upper bound with known prefactors which fall off as inverse of the integer. Hence substituting that into the above one gets,

$$\frac{1}{2\sqrt{3}\pi\left(1 + \frac{r}{3p}\right)} \frac{3^{3p\left(1+\frac{r}{3p}\right)}}{p^2} \sqrt{\frac{1 + \frac{r}{3p}}{1 + \frac{r}{p}}} \frac{\left(1 + \frac{r}{3p}\right)^{3p+r}}{\left(1 + \frac{r}{p}\right)^{p+r}}$$

Since r is bounded and p can grow unboundedly, one can drop powers of $\frac{r}{p}$ and the above expression is asymptotically expected to grow as,

$$\frac{3^{3p}}{2\sqrt{3}\pi p^2}$$

But by definition $p \sim \frac{2(\Delta+j)}{8}$ and hence up to overall constants a lower bound on the asymptotic growth rate of the number of linearly independent states is like,

$$\frac{3^{\frac{3(\Delta+j)}{4}}}{(\Delta + j)^2}$$

Possible generalizations and cautions

From the above argument one might be able to say in general that if one is doing a Polya counting of necklaces which have x_i beads of some colour C_i and the $x_i s$ are determined by solutions of a Diophantine equation of the form, $a_1x_1 + a_2x_2 + ... + a_nx_n = N$ then the asymptotic growth of the number of necklaces is at least,

$$\frac{n^{\frac{nN}{a_1+a_2+\ldots+a_n}}}{N^{\frac{n+1}{2}}}$$

Given the assumptions made in the argument to map the physics context to the above combinatorics problem, it is expected that the above approximation will work best when there are no fermions in the system.

Examples

- One can do a similar analysis as above where there are just two non-commuting scalar fields and each of $\Delta+j = 1$ and then the Diophantine equation one would be looking at is $n + m = \Delta + j$. In this case doing the above analysis would give a lower bound on the asymptotic growth rate of the number of linearly independent states as,

$$\frac{2^{\Delta+j}}{(\Delta+j)^{\frac{3}{2}}}$$

- One can do the same analysis on a theory with only ϕ and D. In that case the Diophantine equation would be $n + 4m = 2(\Delta + j)$ and a lower bound on the asymptotic growth rate of the number of linearly independent states as,

$$\frac{2^{\frac{4(\Delta+j)}{5}}}{(\Delta+j)^{\frac{3}{2}}}$$

As far as I understand the currently known methods of determining this growth rate (or equivalently the "Hagedorn transition temperature") needs one to solve a high order polynomial equation. The order of the polynomial scales with the quantum numbers of the fields involved and the number of fields in the theory.

The above estimation technique reproduces those numbers explicitly and very closely by a calculation of constant complexity. In the method explained here one can very well estimate these numbers by just one calculation (of raising one real number to some power) irrespective of the quantum numbers of the fields or the number of fields.

$Q-$COHOMOLOGY AT EVERY $\Delta + j$

As was earlier noted that the action of Q conserves $\Delta+j$ and hence it maps the vector space of states at every $\Delta+j$ back to itself. So one can write down a matrix for Q in a basis of states chosen at every $\Delta+j$. A natural candidate for such a basis was given in the table earlier which listed the linearly independent Q-closed or Q-not-closed states in the second and third columns. In the previous sections when the objective was to just get a hold on the asymptotic growth of the number of states with $\Delta+j$, one could risk ignoring the fermionic nature of χ but here in the case of calculating the $Q-cohomology$ this can't be done. So some of the states in the table referred to earlier will not be counted since they are 0 if one accounts for the fermionic nature of χ, like $\chi\phi\chi\phi$ at $\Delta+j=4$ and $\phi^2\chi\phi^2\chi$ at $\Delta+j=5$.

To keep things a little general these Q matrices have been calculated with the supersymmetry action corresponding to a superpotential of $\lambda\phi^4$ and hence one would have $Q\chi = \lambda\phi^3$. These Q matrices have been calculated for various values of $\Delta+j$ using the basis (as described above) and all of them square to 0. (as expected from the combinatorial proof given earlier that as operator $Q^2=0$) Hence for these matrices Ker/Im is well defined and for them one can calculate $dim(Ker/Im) = dim(Ker) - dim(Im)$. At any value of $\Delta+j$ this integer would be called the "Q-cohomology" at that $\Delta+j$. At this point it is not clear as to whether there is a completely topological understanding of this number which is being called as the "Q-cohomology".

Some examples of such matrices are as follows,

The matrix of Q at $\Delta+j = \frac{9}{2}$ written in the ordered basis of $\{\phi^9, \phi^6\chi, \phi^3\chi^2, \phi\chi\phi\chi\phi, \chi^3, \phi^4 D\phi, \phi\chi D\phi, \chi\phi D\phi, \phi^2 D\chi, D^2\phi\}$ is a 10×10 matrix having 9 non-zero entries coming from the coefficients of the following 5 non-zero Q actions,

$Q(\phi^6\chi) = \lambda\phi^9$
$Q(\chi^3) = 3\lambda\phi^3\chi^2$
$Q(\phi\chi D\phi) = \phi^3\chi^2 - 3\phi\chi\phi\chi\phi + \lambda\phi^4 D\phi$
$Q(\chi\phi D\phi) = -\phi^3\chi^2 + 3\phi\chi\phi\chi\phi - \lambda\phi^4 D\phi$
$Q(\phi^2 D\chi) = 3\lambda\phi^4 D\phi$

At $\Delta+j = \frac{1}{2}, 1$ the Q matrix is a 1×1 matrix with the only entry being 0 and at say $\Delta+j = \frac{3}{2}$ it is a 2×2 matrix with the basis being $\{\phi^3, \chi\}$ and the only non-zero entry coming from the Q action, $Q(\chi) = \lambda\phi^3$.

It is observed that for all these Q matrices the rank (and hence the dimension of the kernel) is a fixed integer irrespective of the non-zero value of the perturbation parameter λ and it is a different integer at $\lambda = 0$. The values that have been calculated are as follows,

$\Delta+j$	$\lambda \neq 0$ dim(Ker)-dim(Im) = dim(Ker/Im)	$\lambda = 0$ dim(Ker)-dim(Im) = dim(Ker/Im)
5	$7-6=1$	$11-2=9$
$\frac{9}{2}$	$6-4=2$	$9-1=8$
4	$4-2=2$	$6-0=6$
$\frac{7}{2}$	$3-2=1$	$5-0=5$
3	$2-1=1$	$3-0=3$
$\frac{5}{2}$	$2-1=1$	$3-0=3$
2	$1-1=0$	$2-0=2$
$\frac{3}{2}$	$1-1=0$	$2-0=2$
1	$1-1=0$	$1-0=1$
$\frac{1}{2}$	$1-0=1$	$1-0=0$

The number of "Q-cohomology" classes seems to change for the slightest of non-zero values of λ from its values in the free case ($\lambda = 0$) and then it remains constant irrespective of the strength of the deformation. The above can be understood from the fact that in the defining equations for Q the λ can always be scaled to 1 by a variable redefinition of the fields.

It would be great if one could calculate entries in the table as above for arbitrary values of $\Delta + j$ and also be able to map these to the cohomology classes of some manifold which is probably somehow being seen by this theory. If the above pattern continues and these are actually the cohomology classes of some manifold then clearly that manifold is infinite dimensional and it is not so hard to cook up infinite dimensional manifolds which will reproduce the above pattern.

CONCLUSION

The calculations shown in this article are only the beginning of a large conceptual framework that seems to be emerging in recent times about understanding the plethora of superconformal quantum field theories in $2 + 1$ dimensions and super-Chern-Simons-matter theory is a family of interesting candidates among them. Their exceptional properties make them very special and they are also easy to supersymmetrize because of their lack of dynamical gauge degrees of freedom, as was observed in the paper [4] by John Schwartz. As has been said in the introduction, the main objective is to be able to get an analytic control over features of these theories like their BPS spectrum and "Q-cohomology" which are being believed to be independent of the 'tHooft parameter (λ). Added to the above is also the assumption that the "Q-cohomology" calculated in the classical theory is in bijection with the BPS spectrum. Given how important these intuitions are in paving a way towards strong/weak coupling dualities, it becomes extremely important to be able to find a proof or at least stronger evidence in their favour. In the same strain one should point out the need to find a topological understanding of the quantity that is being called "Q-cohomology"

These aims are complicated by the fact that there is increasing evidence for the decrease of the R-charge (from its original value of $\frac{1}{2}$) for the chiral fields as a function of the the 'tHooft parameter. In perturbation theory this was pointed out in the paper [2] by Davide Gaiotto and Xi Yin and in recent months it also has been converted into possibly an exact result in [6] by Daniel Jafferis. These gives strong reasons to believe that the scaling dimensions of the operators will also decrease exactly in the same way so as to preserve the BPS nature of these operators (like $Tr[\phi^n]$). This is in sharp contrast to the premise of this article where all calculations have been done at a fixed value of scaling dimension and R-charge. Hence the challenge is to show that the BPS spectrum remains invariant with the 'tHooft coupling though the scaling dimension and R-charge (the very same quantities defining the BPS condition!) are decreasing.

Here it is appropriate to mention that in the recent work [1] by Shiraz Minwalla, Prithvi Narayan, Tarun Sharma, V. Umesh, Xi Yin, they have shown numerical evidence for the R-charge value to asymptote from above to $\frac{1}{2(\lambda+1)}$. They have also shown that in some theories the R-charge goes to zero as the 'tHooft coupling goes to infinity. That means that in that limit one can add arbitrary number of scalar fields to the operators with no increase in R-charge and this would mean that the theory is developing a continuum of states. This is indicative of there being a description in terms of a theory with a non-compact dimension though there was no such geometrical feature initially! These exciting results need to be understood analytically and hopefully a non-perturbative understanding would eventually emerge.

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APPENDIX-I-DIMENSIONAL REDUCTION TO GET NON-ABELIAN $\mathcal{N}=2$ SUPERSYMMETRY TRANSFORMATIONS IN $2+1$ DIMENSIONS FROM NON-ABELIAN $\mathcal{N}=1$ SUPERSYMMETRY TRANSFORMATIONS IN $3+1$ DIMENSIONS

Let $A, B, ...$ be indices for the gauge group and $\mu, \nu, ...$ be space-time indices. Then the supersymmetric transformation equations with respect to a spinor parameter α for the vector multiplet, $(V_{A\mu}, \lambda_A, D_A)$ are,

$$\delta V_{A\mu} = i\bar{\alpha}\gamma_{4\mu}\lambda_A$$

$$\delta \lambda_A = (-\frac{1}{2}f^4_{A\mu\nu}\gamma_4^{\mu\nu} + D_A\gamma_5)\alpha$$

$$\delta D_A = i\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\lambda_A$$

One renames the third component of the gauge field V_{A3} as F_A and splits the $4-$component fermions into $2-$component spinors in $2+1$ dimensions as,

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

For the spinor components one defines, $\alpha^1 = \alpha_2$ and $\alpha^2 = -\alpha_1$. One substitutes these decompositions in the above set of supersymmetric transformations and sets to 0 all derivatives with respect to the 3^{rd} spatial direction. This converts the supersymmetric transformations into the following set of equations,

$$\delta F_A = i\bar{\alpha}^a\lambda_{Aa}$$

$$\delta D_A = i\bar{\alpha}^a\gamma_3^\mu\partial_\mu\lambda_{Aa}$$

$$\delta V_{A\mu} = i\bar{\alpha}^a\gamma_{3\mu}\lambda_{Aa}$$

$$\delta\lambda_{Aa} = -\frac{1}{2} f^3_{A\mu\nu}\gamma_3^{\mu\nu}\alpha_a + D_A\alpha^a + \gamma_3^\mu D_\mu F_A\alpha^a$$

One notes that the following identity holds in $2+1$ dimensions with any $2-$spinor α,

$$\overline{\gamma_3^\mu\alpha^a} = -\bar{\alpha}^a\gamma_3^\mu$$

This gives,

$$\delta\bar{\lambda}_{Aa} = (\frac{1}{2}\bar{\alpha}_a\epsilon^{\mu\nu\rho}f^3_{A\mu\nu}\gamma_{3\rho} + \bar{\alpha}^a D_A - \bar{\alpha}^a\gamma_3^\mu D_\mu F_A)$$

APPENDIX-II-CONSTRUCTION OF THE $\mathcal{N}=2$ SUPERSYMMETRIC CHERN-SIMON'S THEORY IN $2+1$ DIMENSIONS

In the Abelian case the D_μ will be replaced by just ∂_μ and one can show the following transformation equations,

- $$\delta[FD] = (i\bar{\alpha}^a\lambda_a D - i\bar{\alpha}^a\gamma^\mu\partial_\mu F\lambda_a)$$

- $$\delta(\bar{\lambda}_a\lambda_a) = 2(\frac{1}{2}\bar{\alpha}_a\epsilon^{\mu\nu\rho}\gamma_\mu f^3_{\nu\rho} - \bar{\alpha}^a\gamma^\mu\partial_\mu F + \bar{\alpha}^a D)\lambda_a$$

- $$\delta(\epsilon^{\mu\nu\rho}V_\mu\partial_\nu V_\rho) = i\bar{\alpha}_a\epsilon^{\mu\nu\rho}\gamma_\mu\lambda_a f^3_{\nu\rho}$$

From the above one can observe that the candidate for Lagrangian density is the following term which is kept invariant by the above supersymmetry transformation,

$$\mathcal{L}_{\mathcal{N}=2,Abelian-CS,2+1} = k(\epsilon^{\mu\nu\rho}V_\mu\partial_\nu V_\rho - i\bar{\lambda}_a\lambda_a + 2FD)$$

In the above k (a fixed real number) which is called the Chern-Simons "level". Kao and Lee K1,KL,K2 had shown that unlike in the pure bosonic case or the $\mathcal{N}=1$ case, this k does not undergo any renormalization shift in the $\mathcal{N}=2$ and $\mathcal{N}=3$ case.

The above kind of supersymmetric variation gets harder to calculate for the non-Abelian case and eventually one can show that the similar candidate as above is the term,

$$kTr[\epsilon^{\mu\nu\rho}(V_\mu\partial_\nu V_\rho - \frac{2}{3}V_\mu V_\nu V_\rho) - i\bar{\lambda}_a\lambda_a + 2FD]$$

APPENDIX-III-NOTATION

- The metric being used is $(-1, 1, 1, ..)$
- In $2+1$ dimensional spacetime the gamma matrices are defined as,
  $\gamma_3^0 = -i\sigma^2$, $\gamma_3^1 = \sigma^3$ and $\gamma_3^2 = \sigma^1$
- In the so called "Majorana representation" the $4-$dimensional Gamma matrices can be written such that

$$\gamma_4^i = \begin{bmatrix} & \gamma_3^i \\ \gamma_3^i & \end{bmatrix}$$

(where i runs over $0, 1, 2$)
and

$$\gamma_4^3 = \begin{bmatrix} I & \\ & -I \end{bmatrix}$$

Then γ_5 (only defined in $3+1$ dimensions) is defined to be $\gamma_5 = \gamma_{40}\gamma_{41}\gamma_{42}\gamma_{43}$ and then one has,

$$\gamma_5 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

(where I is a 2×2 identity matrix)

- I am using the following notation for commutators,

$$\gamma_4^{\mu\nu} = \frac{1}{2}[\gamma_4^\mu, \gamma_4^\nu]$$

$$\gamma_3^{\mu\nu} = \frac{1}{2}[\gamma_3^\mu, \gamma_3^\nu]$$

- In $2+1$ dimensions it also holds that,

$$\gamma_3^{\mu\nu} = \epsilon^{\mu\nu\rho}\gamma_{3\rho}$$

- The gauge covariant derivative D acts on a field say ϕ as,

$$(D\phi)_A = \partial_\mu \phi_A + C_{ABC} V_{B\mu} \phi_C$$

where C_{ABC} are the structure constants of gauge group whose lie algebra is spanned by matrices t_A satisfying $[t_B, t_C] = iC_{ABC} t_A$.

- The field strength $f_{A\mu\nu}$ is defined as,

$$f_{A\mu\nu} = \partial_\mu V_{A\nu} - \partial_\nu V_{A\mu} + C_{ABC} V_{B\mu} V_{C\nu}$$

The f will have a superscript of 4 or 3 depending on whether or not the derivatives are being taken in a $3+1$ or a $2+1$ dimensional spacetime.

- The gauge field V_μ, the fermion field λ and the scalar field D are defined as, $V_\mu = t_A V_{A\mu}$, $\lambda = t_A \lambda_A$, $D = t_A D_A$.
- The index a sums over the top and the bottom $2-$spinor components of the $4-$spinor. $a = 1$ is the upper component and $a = 2$ is the lower component.

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