STELLAR EQUILIBRIUM AND CHANDRASHEKHAR'S LIMIT

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Abstract. This is the write-up of a talk that I gave as a part my course evaluation during my BSc. in Physics at Chennai Mathematical Institute (CMI). In this I give a brief overview of the theory of stellar equilibrium with a focus on the Chandrashekhar's Limit.

1. A brief look at the Tolman-Oppenheimer-Volkoff equation

We can think of the star in consideration to be a spherical ball of perfect relativistic fluid in hydrostatic equilibrium which is inducing a metric in its surrounding space-time of the form:

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

(using the conventional Schwarzschild co-ordinates for a static and isotropic space-time)

and the fluid is described by a Stress-Energy tensor of the form:

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)U_\mu U_\nu$$

where U_μ is normalized as $g^{\mu\nu}U_\mu U_\nu = -1$ and since the fluid is at rest we have $U_r = U_\theta = U_\phi = 0$ and hence we have $U_t = -\sqrt{B(r)}$. Further our assumption of hydrostatic equilibrium implies that *Pressure* and *Density* are *not* functions of t but due to spherical symmetry they

Further the condition of Energy-Momentum conservation implies that divergence of $T_{\mu\nu}$ vanishes, which is effectively the condition of the contracted Bianchi Indentity and this gives us the equation:

$$\frac{B'}{B} = -\frac{2p'}{p+\rho}$$

.

Further imposing the Einstein's Equations:

$$R_{\mu\nu} = -8\pi G(T_{\mu\nu} - \frac{g_{\mu\nu}T^\lambda_\lambda}{2})$$

gives us $R_{rr} = -4\pi G(p-\rho)A$, $R_{\theta\theta} = -4\pi G(p-\rho)r^2$ and $R_{tt} = -4\pi G(p+3\rho)B$.

Hence from the above expressions we get:

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$$-r^2\rho'(r) = GM(r)\rho(r)(1+\frac{p(r)}{\rho(r)})(1+\frac{4\pi r^3 p(r)}{M(r)})(1-\frac{2GM(r)}{r})^-1$$

where $M(r) = \int_0^r 4\pi r^2 \rho(r) dr$

A above equation is the **Tolman-Oppenheimer-Volkoff** equation.

We shall use the above equation to its first order of approximation.

2. A glance at Stellar Equilibrium and Stability

I will not give the detailed proofs of the theorem quoted here but will try to give a qualitative picture of the complex processes which are required for equilibrium of the simple fluid ball model of a spherical star.

2.1. Convective Equilibrium.

In a perfect fluid model of the star the most efficient mode of energy transfer is expected to be via convection and in equilibrium the entropy per nucleon must be constant through out the star because otherwise a small amount of fluid containing A nucleons could gain or lose an energy $\frac{A\Delta s}{T}$ when transported from one part of the star to another and convection would therefore disturb the equilibrium.

We also assume that the stars have constant chemical composition throughout and hence pressure can be taken to be a function of $\rho(r)$ without any explicit dependence on r.

2.2. A theorem about stellar stability.

Here we quote without proof a theorem which gives a condition for stellar stability in certain simple conditions.

A particular stellar configuration with uniform entropy per nucleon and chemical compositions will satisfy the Tolman-Oppenheimer-Volkoff equation if and only if the function $M(r)$ is stationary with respect to all variations of $\rho(r)$ that leave unchanged the total number of nucleons in the star which is given by the equation $N = \int 4\pi r^2 n(r)\sqrt{1-\frac{2G(M)(r)}{r}}dr$

We note the following things here :

(1) The variations of ρ considered are such that the perturbed function is still a function only of the radius.
(2) The equilibrium is *stable* if the stationary point is actually a minima.
(3) It can be shown that the condition for equal entropy per nucleon is equivalent to that condition that the thermodynamics of the star is *Polytropic* i.e the Pressure P and the density of the star $\rho(r)$ are related as $P = K\rho^\gamma$ where K and γ are constants. γ is called the *Polytropic Constant* for the star (or equivalently kinetic energy density $= \frac{P}{\gamma-1}$).

2.3. A note.

In this section we have stated some general principles and conditions which can sustain a Polytropic process inside a spherical star. But in the actual analysis of the White Dwarf star we shall make **NO** assumptions about the thermodynamics of the star and from general principles of Theory of Relativity and Quantum Statistics , we shall *derive* the equation of state. This shall also happen to be a Polytropic process!

With this *pre-knowledge* that ultimately our White Dwarf star will be polytropic , in the next section we will define the concept of Mass and Radius of a star which has equilibriated via a polytropic thermodynamics.

3. Defining the radius and mass of a star in a polytropic equilibrium

When the density function of the star is much larger than the pressure function inside the star at all points then one can approximate the Tolman-Oppenheimer-Volkoff equation to the first power of density and hence we get :

Using the same notation as in the previous section.

$$-r^2 \frac{dP}{dr} = GM(r)\rho(r)$$

where $M(r)$ is defined as :

$$Mr = \int_0^r 4\pi r^2 \rho(r) dr$$

Hence combining the above two equations and writing it as purely a differential equation we get :

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G r^2 \rho(r)$$

Further we also need from physical consideration of density and pressure extremize at the center $\rho'(0) = 0$ and $P'(0) = 0$ (where the prime refers to differentiation with respect to the radius.)

So in the above equation we take the case where P and ρ are not any arbitrary functions but are related polytropically as $P = K\rho^\gamma$ (γ being the polytropic constant)

Further we change variables and write the equation as a function of new variables θ and ξ which are related to the original variables via the following equations:

- Let ρ_0 the density of the star at the center. We define ξ through $r = \sqrt{\frac{K\gamma}{4\pi G(\gamma-1)}\rho_o^{\frac{\gamma-2}{2}}}\,\xi$. So we have:
$$\frac{d\xi}{dr} = \sqrt{\frac{4\pi G(\gamma-1)}{K\gamma}\rho_0^{\frac{2-\gamma}{2}}}$$
- We define θ through the equation $\rho = \rho_0 \theta^{\frac{1}{\gamma-1}}$

So in terms of the new variables the equation relating the density and the radius of the star can be put into the form:

$$\frac{1}{\xi^2}\frac{d[\xi^2 \frac{d\theta}{d\xi}]}{d\xi} + \theta^{\frac{1}{\gamma-1}=0}$$

The above is the standard **Lane Emden** function of index $(\gamma-1)^{-1}$.

For ξ near 0 we have the power series solution:

$$\theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{\xi^4}{120(\gamma-1)} - ...$$

We note the crucial property of the **Lane Emden** function that for $\gamma > \frac{6}{5}$ there exists a finite ξ_1 for which $\theta(\xi_1) = 0$ i.e $\rho = 0$ at $\xi = \xi_1$

3.1. **Radius of the star.** So we **define** the radius of the star as that value of the variable r (call that value R) for which density drops to 0. So for $\gamma > \frac{6}{5}$ we can define the radius of the star as:

$$R = \sqrt{\frac{K\gamma}{4\pi G(\gamma-1)}} \rho_0^{\frac{\gamma-2}{2}} \xi_1$$

3.2. **Mass of the star.** Hence it follows from above that most logically we must define the mass of the star as the value of the integral $\int_0^x 4\pi r^2 \rho(r) dr$ till that value of the radius when density drops to 0.Hence we have:

$$M = \int_0^R 4\pi r^2 \rho(r) dr = 4\pi \rho_0^{\frac{3\gamma-4}{2}} (\frac{K\gamma}{4\pi G(\gamma-1)})^{\frac{3}{2}} \int_0^{\xi_1} \xi^2 \theta^{\frac{1}{\gamma-1}}(\xi) d\xi$$

which can also be written as :

$$M = 4\pi \rho_0^{\frac{3\gamma-4}{2}} (\frac{K\gamma}{4\pi G(\gamma-1)})^{\frac{3}{2}} {\xi_1}^2 |\theta'(\xi_1)|$$

4. Special Relativity

Let ϵ be the special-relativistic *kinetic energy* of a particle of linear momentum magnitude p. Then we have ::

$$\epsilon = mc^2[\sqrt{1 + (\frac{p}{mc})^2} - 1]$$

$$\frac{d\epsilon}{dp} = \frac{p}{m\sqrt{1 + (\frac{p}{mc})^2}}$$

5. Statistical Physics for the Star

We assume that the star had ionized to almost a full extent before starting to cool and hence in the cold state we have a large number of free electrons present in the system. But since the mass of the electrons is negligible compared to the masses of the nucleons we assume that the contribution to the mass of the star comes mainly from the nucleons. But since the Fermi Energy is inversely proportional to the mass of the particles we assume that the statistical properties of the star are largely governed by the electrons in the system.

5.1. **Fermi-Dirac Statistics for the electrons.** To analyse the statistics of the star let the variables be :

- N = Total number of electrons in the star
- V = Volume of the star
- n = density of electrons in the star = $\frac{N}{V}$
- m_N = The mass of the nucleons in the star (assuming that there is only one type of them)
- m = mass of an electron.
- μ = number of nucleons of mass m_N per electron (hence it is effectively a measure of the ionization of the star)
- ρ = density of the star. We assume that the nucleons are the predominant contributor to the density and we approximate $\rho \sim n m_N \mu$
- P = Pressure exerted by the electrons in the star.
- e = Kinetic Energy density of the electrons.
- p_F = The Fermi momentum for the star.
- h = Planck's constant.

The electrons can be in 2 possible spin states for each possible value of the momentum state and we assume a coarse graining of the phase-space by cells of volume h^3 (inspired from the Heisenberg's Uncertainty Principle) .Given the above we have ::

$$N = \frac{8\pi V}{h^3} \int_0^{p_F} p^2 dp = \frac{8\pi V p_F^3}{3h^3}$$

Hence we have ::

$$\text{Fermi Momentum for the star } = p_F = (\frac{3n}{8\pi})^{\frac{1}{3}} h = (\frac{3\rho}{8 m_N \mu \pi})^{\frac{1}{3}} h$$

5.2. **Thermodynamics for the star.**

$$\text{Total Kinetic Energy of the electrons} = \frac{8\pi V}{h^3} \int_0^{p_F} mc^2[\sqrt{1 + (\frac{p}{mc})^2} - 1] p^2 dp$$

(Integrating from 0 to the Fermi Momentum the product of energy of a state as a function of linear momentum with the number of states in a differential energy spread about that value)

$$\text{Kinetic Energy density of the electrons} = e = \frac{8\pi}{h^3} \int_0^{p_F} mc^2[\sqrt{1 + (\frac{p}{mc})^2} - 1] p^2 dp$$

Let P= pressure exerted by the electrons in the star.We note that by assumption the electrons are at a temperature of $0K$ and hence ideally all the energy states till the Fermi Energy are fully filled and the higher ones are empty.Since the temperature is OK the notion of *Pressure* exerted by the electrons is a purely Quantum theoretic and it has **no** classical analogue.

We know that $P = \frac{n}{3}\langle p\frac{d\epsilon}{dp}\rangle$

In this system we have::

$$P = \frac{n}{3}\langle\frac{p^2}{m\sqrt{1+(\frac{p}{mc})^2}}\rangle$$

So explicitly we have ::

$$P = \frac{N}{3V}\frac{\text{Total}\{\frac{p^2}{m\sqrt{1+(\frac{p}{mc})^2}}\}}{N}$$

Hence

$$P = \frac{8\pi}{3h^3}\int_0^{p_F}\frac{p^4}{m\sqrt{1+(\frac{p}{mc})^2}}dp$$

and we also have

$$e = \frac{8\pi}{h^3}\int_0^{p_F} mc^2[\sqrt{1+(\frac{p}{mc})^2}-1]p^2dp$$

5.3. **Critical Density.** We define ρ_c as the Critical Density of the star at which

Fermi Energy = Rest Mass Energy of the Electron

i.e

$$\frac{p_F^2}{2m} = mc^2$$

which gives us

$$\rho_c = \frac{2^{\frac{9}{2}}m_N\mu m^3c^3}{3h^3}$$

5.4. **The 2 important regions of analysis.** Hence using the value of ρ_c as a benchmark we realize that the star is expected to show different kinds of behaviour in the following two regions of values :

(1) When $\rho \ll \rho_c$ which implies that $p_F \ll \sqrt{2}mc$
(2) When $\rho \gg \rho_c$ which implies that $p_F \gg \sqrt{2}mc$

6. The results in the 2 cases

6.1. Results when $p_F \ll \sqrt{2}mc$. We quote the final results in this case:

- $e = \frac{3P}{2}$ and $P = K\rho^{\frac{5}{3}}$ where $K = \frac{\hbar^2}{15m\pi^2}(\frac{3\pi^2}{m_N\mu})^{\frac{5}{3}}$
- So $\gamma = \frac{5}{3}$ and from the standard tables we have $\xi_1 = 3.65375$ and $-\xi_1^2\theta'(\xi_1) = 2.71406$
- $M = \frac{1}{2}\sqrt{\frac{3\pi}{8}}(2.71406)(\frac{\hbar^{\frac{3}{2}}c^{\frac{3}{2}}}{m_N^2\mu^2G^{\frac{3}{2}}})\sqrt{(\frac{\rho_0}{\rho_c})} = \frac{2.79}{\mu^2}\sqrt{(\frac{\rho_0}{\rho_c})}M_\odot$
- $R = \sqrt{\frac{3\pi}{8}}(3.65375)(\frac{\hbar^{\frac{3}{2}}}{c^{\frac{1}{2}}G^{\frac{1}{2}}mm_N\mu})(\frac{\rho_0}{\rho_c})^{-\frac{1}{6}} = \frac{2\times10^4}{\mu}(\frac{\rho_0}{\rho_c})^{-\frac{1}{6}}Km$

In this **non-relativistic regime** most important thing to note is that the mass of the star can be arbitrarily high depending on the central density of the star (assuming that as we increase the density no other atomic and nuclear processes start off than the already present convection of the fluid that maintains the thermal equilibrium)

6.2. Results when $p_F \gg \sqrt{2}mc$. We quote the final results in this case:

- $e = 3P$ and $P = K\rho^{\frac{4}{3}}$ where $K = \frac{\hbar}{12\pi^2}(\frac{3\pi^2\rho}{m_N\mu})^{\frac{4}{3}}$
- So $\gamma = \frac{4}{3}$ and from the standard tables we have $\xi_1 = 6.89685$ and $-\xi_1^2\theta'(\xi_1) = 2.01824$
- $M = \frac{1}{2}\sqrt{3\pi}(2.01824)(\frac{\hbar^{\frac{3}{2}}c^{\frac{3}{2}}}{m_N^2\mu^2G^{\frac{3}{2}}}) = \frac{5.87}{\mu^2}M_\odot$
- $R = \frac{1}{2}\sqrt{3\pi}(6.89685)(\frac{\hbar^{\frac{3}{2}}}{c^{\frac{1}{2}}G^{\frac{1}{2}}mm_N\mu})(\frac{\rho_c}{\rho_o})^{\frac{1}{3}} = \frac{5.3\times10^4}{\mu}(\frac{\rho_c}{\rho_o})^{\frac{1}{3}}Km$

Here in the **extreme-relativistic regime** the fascinating thing to observe is that the *allowed* mass of the star in this fluid model is a precise number which depends **ONLY** on the ionization of the fluid and **NOT** on the density and other thermodynamic parameters of the fluid.

This special value of the mass of the star is what is called the **Chandrashekhar's Limit** (derived here the so called White Dwarf Stars).

7. A sketch of the approximation technique

To get the above limits we have to effectively take limits on the expressions for Pressure and Energy that were stated earlier. We had

$$P = \frac{8\pi}{3h^3}\int_0^{p_F}\frac{p^4}{m\sqrt{1+(\frac{p}{mc})^2}}dp$$

On this integration we do the following change of variables:

$p = mcsinh\theta$ and hence we define $p_F = mcsinh\theta_F$.

With this the integration transforms to:

$$P = \frac{8\pi c^5m^4}{3h^3}\int_0^{\theta_F} sinh^4\theta d\theta$$

7.1. **A standard result.** In a paper in 1942 *Kothari and Singh* showed the following result:

$$\int_0^{\theta_F} sinh^4\theta d\theta = \frac{A(x)}{8}$$

where

$$A(x) = x\sqrt{x^2+1}(2x^2-3) + 3sinh^{-1}x$$

Then they showed the following asymptotic expressions:

$$A(x) = \frac{8}{5}x^5 - \frac{4}{7}x^7 + \frac{x^9}{3} - \frac{5x^{11}}{22} + ...(\text{if } x \ll 1)$$

and

$$A(x) = 2x^4 - 2x^2 + 3(ln(2x) - \frac{7}{12}) + \frac{5}{4x^2} + ...(\text{if } x \gg 1)$$

7.2. **Back to the approximating the Pressure function ....** Hence using the above result we can write that :

$$P = \frac{8\pi c^5 m^4}{3h^3}\int_0^{\theta_F} sinh^4\theta d\theta = \frac{8m^4\pi c^5}{3h^3}\frac{A(\frac{p_F}{mc})}{8}$$

Hence using the Kothari-Singh expansion to first order we have the following expressions:

$$\text{when } p_F \ll \sqrt{2}mc \text{ we have } P \sim \frac{8\pi}{15h^3}\frac{p_F^5}{m}$$

and

$$\text{when } p_F \gg \sqrt{2}mc \text{ we have } P \sim \frac{8\pi c}{3h^3}p_F^4$$

7.3. **Illustrating the Approximating the Energy function in one of the cases.** The process of approximating the Energy function is similar for a large part for the two different limits that we are concerned with. We show the method for the case of $p_F \ll mc$.

Given the expression:

$$e = \frac{8\pi}{h^3}\int_0^{p_F} mc^2[\sqrt{1+(\frac{p}{mc})^2} - 1]p^2dp$$

We can integrate the above by parts to get:

$$e = \frac{8\pi mc^2 p_F^3}{3h^3}(\sqrt{1+(\frac{p_F}{mc})^2} - 1) - \frac{8\pi}{3h^3 m}\int_0^{p_F}\frac{p^4dp}{\sqrt{1+(\frac{p}{mc})^2}}$$

We recognise that the second term is precisely our earlier expression for P.

The first term can be expanded in a binomial series for the case $p_F \ll mc$ to get $\frac{5}{2}$ of the first term for the expansion of P in the *Kothari-Singh* series.

Hence we get to the first order of approximation that:

$$e = \frac{8\pi mc^2 p_F^3}{3h^3}(\frac{p_F^2}{2m^2c^2}) - P = \frac{5P}{2} - P = \frac{3P}{2}$$

So when $p_F \ll mc$ star will follow polytropic thermodynamics with a polytropic constant of $\gamma = \frac{5}{3}$.

In the other limit too similar methods can be used to derive the polytropic constant which is the only variable which enters the expression for Mass and Radius of the star.

8. Final Remark

At the end of this analysis we see that there exists a special value of mass of the star, *independent of the thermodynamic parameters* and equal to $\frac{5.87}{\mu^2} M_\odot$ which the star must have if it its thermodynamics is dominated by highly relativistic electrons at zero temperature.

The intriguing point to note is that the Chandrashekhar's Limit is a very special number defined by fundamental constants of the universe defined through diverse aspects of Physics like from quantum theory comes the Planck's constant and those coming from special relativity like the speed of light and from gravitational physics, the Gravitational Constant and the mass of the nucleons.

9. Acknowledgement

Chandrashekhar's limit was the first concept in physics that attracted me to the subject when as a kid I read the "Brief History of Time" by Stephen Hawking. I would like to express my deep gratitude for the help received from Prof.Ghanshyam Date during the proces of studying General Relativity and for his extremely detailed course. I got immense help from him even after the course when during another project in the next semester I was strying to simulate on the computer orbits of test-particles in the Schwarzchild Metric. I would also like to thank Prof.G.Rajasekaran who cleared various of my doubts when I first started studying General Relativity during my first year of BSc.(Hons.). He painstakingly explained to me many of the subtle arguments of Landau and Lifshitz in their chapters on General Relativity. Much of the core content of this seminar was taken from the books by Weinberg and Pathria.