LIFTING MAPS

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Abstract. Once geometric questions have been translated in the language of spaces and maps one faces the generic situation of trying to build up maps between new spaces starting with simple maps between familiar spaces. This process almost always cleans the situation of extraneous information and lets us focus on the most important geometric features.

In this article I will try to discuss some of the important general techniques that exist for building maps. There is a brief discussion of the Universal Mapping Question. The article builds up towards understanding the question of lifting of maps and its ramifications in topics like existence of **Global Cross-Sections** and the **Path Lifting Theorem** (these two for locally trivial bundles) and **Unique Lifting Theorem** (for covering spaces).

While discussing these topics at times some exciting notions of Physics have been pointed out which have their modest beginnings in these concepts.

1. Introduction

Through systematic techniques of inducing topologies like Quotient Topology, Relative Topology and Product Topology one can create and topologize complicated spaces starting from simple spaces. Once we have a huge variety of spaces with us and the obvious maps between some of them the bigger aim is to be able to bunch these together into triplets forming **Commutative Diagrams** which have been seen to be very compact methods to store geometric information.

Further it is extremely satisfying when multiple commutative diagrams can be joined together consistently to give larger and larger such diagrams. An important example of when such a thing happens is in the study of maps between projective spaces and spheres.

Another important structure to succinctly store information about inter-relationships between spaces is through **Locally Trivial Bundle** which shall also be explained below.

1.1. **Commutative Diagram.** A **Commutative Diagram** can be defined in any category using 3 Objects (here Topological Spaces) and 3 Morphisms (here Maps between Topological Spaces) between them but for the current purpose I keep to the following definition:

A **Commutative Diagram** is a set of 3 topological spaces (say X,Y and Z) and 3 maps (say $f : Y \to Z$,$g : X \to Y$ and $h : X \to Z$) such that $f \circ g = h$

In practical purposes the need might be to work in narrower categories like those of Topological Spaces where the only allowed morphisms are homeomorphisms or continuous maps.

In what follows unless otherwise stated X, Y and Z shall label topological spaces.

1.2. **Locally Trivial Bundle.** The structure of a **Locally Trivial Bundle** is a very generic structure in geometry and Physics that comes up in almost any interesting geometrical question (like in Quantum Hall Effect) and hence one would like to put down its definition.

A **Locally Trivial Bundle** *consist of* 3 *topological spaces namely* E *(the* **Total Space** *or* **Bundle Space***), a Hausdorff space* X *(the* **Base Space***) and another Hausdorff space* Y *(the* **Fiber Space***) and a continuous surjective map* π *(called the* **Projection Map***) from* E *to* X *such that for each* $x_0 \in X$ *there exists an open set* $V \in X$ *containing* x_0 *and a homeomorphism* $\phi : V \times Y \to \pi^{-1}(V)$ *such that* $\pi \circ \phi(x, y) = x$ *for all* $(x, y) \in V \times Y$*.*
V *is called the* **Local Trivializing Neighbourhood** *of* x_0 *in* X*.*

One refers to the above structure with the symbol (E, π, X, Y).

Two common examples of Locally Trivial Bundle are the Tangent Bundle (and its tensor powers) of differential manifolds (which is more specifically a **Vector/Tensor Bundle** where the fibers are also vector/tensor spaces) and Hopf Fibrations (for the complex and the quaternionic Hopf Bundle it is more specifically a **Principle Bundle** where the fibers are Lie groups).

We note the following things:

- One can show that the above definition implies that π is an open map and E is Hausdorff.
- If E is connected and Y is a discrete space then the Locally Trivial Bundle is more specifically called a **Covering Space**. As an example one notes that the real Hopf Bundle is a Covering Space whereas the complex and the quaternionic Hopf Bundles are Principle Bundles but are not Covering Spaces.

We note the following further things about Covering Spaces:

- For a Covering Space the "Locally Trivializing Neighbourhood" of a point in the Base Space is more specifically called an "Evenly Covered Neighbourhood" of that point.
- The inverse image of any evenly covered neighbourhood under the inverse of the Projection Map is a disjoint union of open sets. Each of these open sets is called a **Sheet** over the evenly covered neighbourhood.
- From the connectedness of the Total Space one can show that the cardinality of the fibers over any point in the Base Space is the same.

2. The kinds of questions

Once the need is recognized to organize geometric information in the form of commutative diagrams the following 4 kinds of operations need to be done while constructing these commutative diagrams:

It must be noted that while doing the following operations there is almost always some criteria to be fulfilled than just the bare minimum conditions said below and that criteria is what makes the job non-trivial otherwise almost always there exists some trivial solution of no socially redeemable value. The most common demand is that all the initial maps are continuous (resp. homeomorphism) and at the end of the operations all the final maps should also be continuous (resp. homeomorphism).

- **Extend maps** from subspaces to the full space or vice-versa.
- **Factor maps** like given a continuous map from X to Y and a topological space Z to find (or prove the non-existence!) of continuous maps from X to Z and Z to Y such that they form a commutative diagram.
- **Quotient a map** i.e given a continuous map between X and Y to try to push it down to a continuous map from some quotient space of X defined by some Quotient Map as defined below:
  If $Q : X \to Z$ is a surjective map then Z can be given the Quotient Topology through Q by defining those sets of Z to be open whose inverse image in X is open in the topology on X. Then $Q : X \to Z$ is called the Quotient Map
- **Complete a commutative diagram** i.e given X, Y and Z and any 2 of the maps f,g and h (using the notation of section 1.1) to either prove the existence (and construct) or prove the non-existence of the third map to complete the commutative diagram.

2.1. **A Remark.** Many times when the question is about completing the commutative diagram and the maps are to be homeomorphisms and the spaces are simple enough one can prove (if true!) the non-existence of a completion by looking at the homology groups of the 3 spaces since we know that if 2 spaces have non-isomorphic homology groups then they are non-homeomorphic.

3. Spotlight on completing the commutative diagram

Of the 4 generic kinds of constructions with maps the one that is of paramount interest is that of completing the commutative diagram and almost always one sees that the other 3 kinds act as intermediates to achieve this final goal. Hence the 2 possible kinds of variations of completing the commutative diagram have been distinguished by 2 special names as follows:

3.1. Lifting Of Maps.

Since the most common situation arises with continuous maps, for what follows I drop the generalization of using any arbitrary map.

Lifting Of Map is the question of finding a continuous map $g : X \to Y$ such that the commutative diagram is completed when continuous maps $h : X \to Z$ and $f : Y \to Z$ are given beforehand.
We will say that the continuous map from X to Z has been lifted to a continuous map from X to Y.

Immediately 2 important things should be noted:

- Not always is such a lift possible as will be soon demonstrated.
- Any path in the base space of a locally trivial bundle can be lifted to the total space and something similar also hold for covering spaces. (Actually what is true is stronger than what is stated in the last statement) These will be made precise in the following 2 important theorems **Path Lifting Theorem** and the **Unique Lifting Theorem** to be explained in the 3^{rd} part of these articles.

3.2. Universal Mapping Question.

This is the question of finding a continuous map $f : Y \to Z$ to complete the commutative diagram when continuous maps $g : X \to Y$ and $h : X \to Z$ are given.
If such an f is found then one says that $g : X \to Y$ is the *solution* of an Universal Mapping Question.

In this article I shall not dwell on this issue but would only like to point out that solutions to the Universal Mapping Question provides a very elegant method of defining various structures like Tensors.

4. Important Results

Now that the general structure and definitions has been set up, over the coming sections let me list some of the results about the 4 mapping constructions I had discussed earlier namely **Extend maps**, **Factor maps**, **Quotient a map**, **Complete a commutative diagram**.

I have chosen the results based on their fundamentality and attractive generality.

I am arranging the results in increasing order of complexity for logical coherence.

5. About extending maps

I consider the following 2 theorems about extending maps:

5.1. Shrinking the Domain.

The following result holds:

Theorem 1. *Let X' be a subspace of X and $f : X \to Y$ be a continuous map. Then the restriction $f \mid X' : X' \to Y$ is continuous. In particular the inclusion map $i : X' \hookrightarrow X$ defined by $i(x) = x$ $(\forall x \in X')$ is continuous.*

Proof. Let U be open in Y. Since f is continuous $f^{-1}(U)$ is open in X and therefore $X' \cap f^{-1}(U)$ is open in X' (by the very definition of "relative topology"). But $X' \cap f^{-1}(U) = (f \mid X')^{-1}(U)$. So $(f \mid X')^{-1}(U)$ is open in X' and hence $f \mid X'$ is continuous. The inclusion map is continuous since it is just a special case of the above where the original map is the identity mapping of X which is obviously continuous. □

5.2. Shrinking and Enlarging the Range.

The following result holds:

Theorem 2. *Let Y' be a subspace of Y. Let $f : X \to Y$ be a continuous map such that $f(X) \subseteq Y'$ then regarded as a map into Y', $f : X \to Y'$ is continuous. Conversely given a continuous map $g : X \to Y'$ it is also continuous regarded as a map into Y as $f : X \to Y$.*

The above follows trivially from the definition of "relative topology"

5.2.1. *Few Remarks.*

One should take note of the essential asymmetry between the domain and the range as it manifests itself through the above two theorems. One can easily shrink as well as enlarge the range but can only shrink the domain with ease. The question of enlarging the domain is much more profound and some of whose aspects I plan to bring forth over the next few sections.

In retrospect one can say that this asymmetry was expected given how the notion of continuity is defined.

The above 2 theorems actually relieve us of a lot of complications since many interesting topological spaces (like spheres) occur as subspaces of $\mathbb{R}^n$ and it is generally easy to check continuity of a map if it is between Euclidean spaces . Hence the above two theorems let us extend these simple to check continuous maps to continuous maps between subspaces of Euclidean spaces provided they have been topologized with the relative topology (which is usually the case).

As an immediate example I would like to mention that while giving stereographic projections of a sphere (a very important map for understanding the manifold structure of the sphere under the relative topology of the ambient Euclidean space) the the issue of proving the continuity of the stereographic projection would have been very complicated without the above theorems whereas continuity is almost trivial to prove when thought of as between Euclidean spaces.

The above also demonstrates the most important raison de etre for the definition of relative topology!

5.3. **The Glueing Lemma.** This I feel is a very important example of a very simple yet extremely powerful theorem:

Theorem 3. *Let X and Y be topological spaces and assume that $X = A_1 \cup A_2$ where A_1 and A_2 are open sets in X. Suppose $f_1 : A_1 \to Y$ and $f_2 : A_2 \to Y$ are continuous such that $f_1 \mid A_1 \cap A_2 = f_2 \mid A_1 \cap A_2$. Then the map $f : X \to Y$ is continuous which is defined as:*

$$f(x) = \begin{cases} f_1(x), & x \in A_1 \\ f_2(x), & x \in A_2 \end{cases}$$

Proof. The result is trivial if $A_1 \cap A_2$ is empty. Hence let us assume that it is not so. Then note that f is well-defined since for $x \in A_1 \cap A_2$ implies that $f_1(x) = f_2(x)$. Let V be an open set in Y. Then

$$f^{-1}(V) = f^{-1}(V) \cap X = f^{-1}(V) \cap (A_1 \cup A_2) = [f^{-1}(V) \cap A_1] \cup [f^{-1}(V) \cap A_2] = f_1^{-1}(V) \cup f_2^{-1}(V)$$

Since f_1 is continuous $f_1^{-1}(V)$ is open in A_1 and hence open in X and similarly $f_2^{-1}(V)$ is open in X. Hence $f_1^{-1}(V) \cup f_2^{-1}(V)$ is open in X. So f is continuous. □

The result is also valid if A_1 and A_2 are both closed. Then one needs to use the equivalent definition of continuous mapping in terms of closed sets.

5.3.1. *Few Remarks.* The Glueing Lemma no matter how obvious it might seem is an answer to the question of enlarging the domain of a map, to which motivations were given earlier, albeit in a very simple scenario with weak constraints.

The above theorem gives us a clear and simple way to patch together continuous maps to form continuous maps over larger spaces. This is of deep importance when we are dealing with manifolds where the local geometry is simple to see and hence locally continuous maps are easy to see and then the above theorem lets us easily patch together these maps.

A very notable use of The Glueing Lemma is in establishing the homeomorphism between certain projective spaces and certain spheres like ($\mathbb{RP}^1 \cong S^1$)

The process of patching together of local maps might soon run into inconsistencies due to the large scale geometry of the domain space like in the folk lore example of an annulus. Hence a more complicated question is to ask as to how far can this process of extending be continued. This is no more a question about local patching like this Glueing Lemma but a question about the global topology of the space. These questions can be answered by more sophisticated tools like those of Cohomology or Sheaf Theory. The very notion of Etale Spaces (in Sheaf Theory) is so defined so as to provide an elegant reformulation of this question of global extension of local maps.

Hence this Glueing Lemma can be seen to be a modest beginning towards a large and exciting arena of questions.

6. About Quotienting Maps

I think the following theorem is a very powerful statement about pushing down maps to quotient spaces maintaining their continuity:

Theorem 4. *Let $Q : X \to Y$ be a quotient map and $f : X \to Z$ a continuous map with the property that $f \mid Q^{-1}(y)$ is a constant map for each $y \in Y$. Then there exists a unique continuous map $f' : Y \to Z$ such that $f' \circ Q = f$*

Proof. For each $y \in Y$ we define $f'(y) = f(x)$ for any $x \in Q^{-1}(y)$. f' is well defined because f is constant on the fibers of Q. Obviously for all $x \in X$, $(f' \circ Q)(x) = f'(Q(x)) = f(x)$ so $f' \circ Q = f$.

Now we need to show that f' so defined is continuous. Let U be an open set in Z and we need to show that $f'^{-1}(U)$ is open in Y but that can be true if $Q^{-1}(f'^{-1}(U))$ is open (definition of relative topology) in X but $Q^{-1}(f'^{-1}(U)) = (f' \circ Q)^{-1}(U) = f^{-1}(U)$ but $f^{-1}(U)$ is open in X since f is given to be continuous.

To prove uniqueness let $f'' : Y \to Z$ also satisfy $f'' \circ Q = f$. Then for each $x \in X$ we have $f''(Q(x)) = f'(Q(x))$. If $f'' \neq f'$ then there exists some $y \in Y$ such that $f''(y) \neq f'(y)$ but since Q is surjective for any $y \in Y$ there is $x \in X$ such that $Q(x) = y$. So there will exist $x \in X$ such that $f''(Q(x)) \neq f'(Q(x))$, but that is absurd by the very definition of f'' and f'. Hence $f'' = f'$ and uniqueness is proven. ☐

6.1. **Few Remarks.** The sheer generality of the premises on which the above theorem holds makes it very powerful. The above theorem will have immense utility in the theory of locally trivial fiber bundles. Locally trivial bundles show up up ever so often in almost any geometric question. To cite two interesting areas in physics where this appears is in the theory of magnetic monopoles and Quantum Hall Effect.

The above theorem lets us push down any continuous map on the total space to a continuous map on the base space. One should note that although the quotient space of the total space of a locally trivial bundle under identification of fibers is homeomorphic to the base space, the homeomorphisms of the total total space only go down as continuous maps from the base spaces since the quotienting map is just a continuous map.

We note the following ramifications of this theorem about quotient maps:

- The above theorem also lets us safely push down the well-known maps from the spheres to continuous maps from the projective spaces which are less intuitive.
- This theorem is also crucially needed to formally prove the otherwise intuitive fact that the suspension of S^{n-1} is homeomorphic to S^n.
- The following is a hiked up version of the theorem just proved and its proof though might not be very obvious but is very intuitive in the light of the above theorem.

  Let G be a topological group and X a topological space and $(g, x) \to g.x$ a transitive left action of G on X. Fix $x_0 \in X$, let $H = \{g \in G : g.x_0 = x_0\}$ be its isotropy subgroup and define $Q' : G \to X$ by $Q'(g) = g.x_0$. Then H is a closed subgroup of G and $Q : G \to G/H$ be the canonical projection, then there exists a unique continuous bijection $\phi : G/H \to X$ for which the diagram commutes i.e $\phi \circ Q = Q'$. Further if either G is compact or Q' is an open map then ϕ is a homeomorphism.

  The above theorem provides an elegant way to unify the study of higher dimensional spheres (S^n(s)) and the classical Lie groups i.e the orthogonal, unitary and symplectic groups.

7. Not all maps can be lifted

Just after defining the notion of **Lifting Of Maps**, I had mentioned that not all maps can be lifted. I feel that it is very exciting to see an actual example of a situation where it can fail.

One must note that in almost any example that can be given for a situation where a certain map cannot be lifted the conclusion can be almost trivially gotten by looking at the Homology Groups of the spaces if the spaces are simple enough. But here in this pretty simple and standard example we shall get the conclusion using elementary point-set topology.

Let $P : \mathbb{R}^1 \to S^1$ be the continuous maps $P(x) = e^{i2\pi x}$ whose fiber over any point in S^1 is $\mathbb{Z}$. Let $id : S^1 \to S^1$ be the continuous identity mapping of the circle. Now the question is to lift this identity map to a continuous map $f : S^1 \to \mathbb{R}$ such that $P \circ f = id$.

It can be shown that such a lift does not exist.

We recall the following facts:

- That continuous image of a compact (resp. connected) space is compact (resp. connected)
- S^1 is compact and connected
- The Heine-Borel theorem (that closed and bounded subspaces of Euclidean spaces are the only compact subspaces of them)
- The fact that the only connected subspaces of $\mathbb{R}$ are the intervals.

From the above we conclude that if the required lift exists then there exists $a, b \in \mathbb{R}$ and $a < b$ such that $f(S^1) = [a, b]$. Further this map cannot be one-to-one since that would give a homeomorphism from S^1 to $[a, b]$ which is impossible by the usual connectedness argument. So there exists points $y_0, y_1 \in S^1$ such that $y_0 \neq y_1$ but $f(y_0) = f(y_1)$. But by definition $P \circ f(y_0) = P \circ f(y_1)$ which means that $id(y_0) = id(y_1)$ and hence $y_0 = y_1$. Hence a contradiction.

Hence we have shown that the lift sought for doesn't exist.

The above example is a special case of a larger structure as defined below:

7.1. **Cross-Section.** One defines a **(Global Cross-Section** of a locally trivial bundle (E, π, X, Y) to be a lift to E of the identity map $id : X \to X$ i.e a continuous map $s : X \to E$ such that $\pi \circ s = id$.

Intuitively a global cross-section is a continuous selection of an element from each fiber $\pi^{-1}(x)$ for all $x \in X$. One can see that a vector field on a manifold forms a simple example of a global cross-section of the tangent-bundle of a differential manifold.

If the above construction is done by replacing X by some open-set (say U) of X then one calls it a **Local Cross-Section** over U.

It is clear that not all locally trivial bundles have a global cross-section as shown in the above example for the bundle $(\mathbb{R}, P, S^1, \mathbb{Z})$.

The concept of cross-section forms a natural language to state various geometrical properties which would be otherwise very cumbersome to enunciate.

In the following sections what shall be explained are the two important positive results about lifting of maps mentioned earlier namely Path Lifting Theorem and Unique Lifting Theorem. To build up towards them we need the following concept about compact subspaces of $\mathbb{R}^n$.

8. Lebesgue Number

Let $A \subseteq \mathbb{R}^n$ and $\mathbb{R}^n$ be equipped with the natural Euclidean Norm. Then we define **Diameter of** A (denoted as **diam(A)**) for $A \neq \phi$

$$diam(A) = sup\{\|y - x\| : x, y \in A\}$$

If $A = \phi$ then we define $diam(A) = 0$.

Let X be a subspace of $\mathbb{R}^n$ then $U_r(x, X) = \{y \in X : \|y - x\| < r\}$. Then one can trivially see from triangle inequality that if $A \subseteq X$ and $diam(A) < r$ and $x \in A$ then $A \subseteq U_r(x, X)$.

Now we have the following important notion:

*L*et X be a compact subspace of $\mathbb{R}^n$. Then for each open-cover Γ there exists a positive number $\lambda(\Gamma)$ called the **Lebesgue Number for** Γ with the property that any $A \subseteq X$ with $diam(A) < \lambda$ is entirely contained in some element of Γ.

For each $x \in X$ choose $r(x) > 0$ so that $U_{r(x)}(x, X)$ is contained in some element of Γ. Then $\{U_{\frac{r(x)}{2}} : x \in X\}$ is an open cover for X. Let the finite sub-cover extractible from this open-cover be $\{U_{\frac{r(x_1)}{2}}(x_1, X), ..., U_{\frac{r(x_k)}{2}}(x_k, X)\}$. Now the claim is that $\lambda = min\{\frac{r(x_1)}{2}, ..., \frac{r(x_k)}{2}\}$. To prove that this λ is the Lebesgue Number we need to show that every $U_\lambda(x, X)$ for all $x \in X$ is contained in some element of Γ. Now every $x \in X$ is contained in some $U_{\frac{r(x_i)}{2}}(x_i, X)$, so for any $y \in U_\lambda(x, X)$,

$$\|y - x_i\| \leq \|y - x\| + \|x - x_i\| < \lambda + \frac{r(x_i)}{2} \leq r(x_i)$$

So $U_\lambda(x, X) \subseteq U_{r(x_i)}(x_i, X)$. But $U_{r(x_i)}(x_i, X)$ is contained in some element of Γ and hence the proof.

9. Path Lifting Theorem

In whatever follows whenever the term "path" is used the map implied from $[0,1]$ is continuous although it shall not be explicitly stated.

The Path Lifting Theorem is an extremely important positive statement about lifting of maps in a certain case. It effectively guarantees that in a Locally Trivial Bundle one can always lift a path in the Base Space to a path in the Total Space. More precisely:

Theorem 5. *Let (E, π, X, Y) be a Locally Trivial Bundle and $\alpha; [0,1] \to X$ be a path in the Base Space X. Then for any p in the fiber $\pi^{-1}(\alpha(0))$ there exists a lifted path $\tilde{\alpha} : [0,1] \to E$ such that $\pi \circ \tilde{\alpha} = \alpha$ and $\tilde{\alpha}(0) = p$.*

Proof. The idea is to lift α inductively in small parts and these parts will always be small enough to use the local trivialization of the bundle.

The choice of these small parts is done in the following way:

We subdivide the interval $[0,1]$ into sub-intervals with endpoints $0 = s_0 < s_1 < s_2 ... < s_{n-1} < s_n = 1$ in such a way that $\alpha([s_{i-1}, s_i])$ for $i = 1, 2, ..., n$ is contained in some local trivializing neighbourhood V_i in X. To do this cover X with locally trivializing neighborhoods V and consider the open cover of $[0,1]$ formed by the corresponding $\alpha^{-1}(V)$. Let λ be the Lebesgue Number of this open cover and choose a positive integer n large enough so that $\frac{1}{n} < \lambda$. Then select $s_i = \frac{i}{n}$ for $i = 1, 2, ..., n$.

Now we show by induction that for each $i = 1, 2, ..., n$ there exists a continuous map $\alpha_i : [0, s_i] \to E$ such that $\alpha_i(0) = p$ and $\pi \circ \alpha_i = \alpha|[0, s_i]$. We start the induction with $\alpha_0(0) = p$ and α_n will be the required lifted path $\tilde{\alpha}$.

Suppose $0 \leq k \leq n$ and that we have already lifted till $\alpha_k : [0, s_k] \to E$ such that as required $\alpha_k(0) = p$ and $\pi \circ \alpha_k = \alpha|[0, s_k]$.

Then $\alpha([s_k, s_{k+1}])$ is contained in some locally trivializing neighbourhood V_k of X. Let $\phi_k : V_k \times Y \to \pi^{-1}(V_k)$ be a homomorphism satisfying $\pi \circ \phi_k(x, y) = x$ for $(x, y) \in V_k \times Y$.

ϕ_k is what will extend α_k to α_{k+1}. So we need to check that they match at their common end.

So $\pi \circ \alpha_k(s_k) = \alpha(s_k)$ (by definition of α_k). Hence $\alpha_k(s_k) \in \pi^{-1}(\alpha(s_k)) \subseteq \pi^{-1}(V_k)$. Thus $\phi_k^{-1}(\alpha_k(s_k)) \in V_k \times Y$ and so $\phi_k^{-1}(\alpha_k(s_k)) = (\alpha(s_k), y_0)$ for some $y_0 \in Y$. So we have $\phi_k(\alpha(s_k), y_0) = \alpha_k(s_k)$.

Now define $\alpha_{k+1} : [0, s_{k+1}] \to E$ as :

$$\alpha_{k+1}(s) = \begin{cases} \alpha_k(s), & s \in [0, s_k] \\ \phi_k(\alpha(s), y_0), & s \in [s_k, s_{k+1}] \end{cases}$$

The Glueing Lemma ensures that α_{k+1} is continuous.

Moreover $\alpha_{k+1}(0) = \alpha(0) = p$ and $\pi \circ \alpha_{k+1} = \alpha|[0, s_{k+1}]$.

Hence induction follows and the proof is complete. □

9.1. **Few Remarks.** We note the following important things in the light of the Path Lifting Theorem:

- From the Path Lifting Theorem it trivially follows that in a Locally Trivial Bundle if the Base Space and the Fiber are path connected then the total space is also path connected.

  Given any two points in the total space we can project it down and path connect it in the Base Space and then lift up this path by the above theorem so that the initial point is one of the given points. The lifted path is not guaranteed to end in the other point but it is sure to end in another point in the same fiber as as the other point. Now since the fiber is path connected we can join them up. Again the Glueing Lemma will ensure continuity.
- We note that in the above proof that by varying y_0 we can arbitrarily "vertically" distort the lifted path keeping all conditions satisfied. There is no guarantee of uniqueness about the lifted path. The whole idea of when is the lifted path unique and what conditions to specify to get an unique path lift is central to all of "Gauge Field Theory" in Physics and that "choosing a gauge" and the notions of "gauge potentials" are ways of specifying unique path lifts and correspond to the idea of choosing special kinds of of "connections" from their moduli spaces on certain Principle Bundles with the fiber being the Gauge Group (which is a Lie Group).
- A relatively simple case of when one can control this meandering of the lifted path is with paths on a **Covering Space** as explained in the following theorem.

10. Unique Lifting Theorem

Since a Covering Space is a Locally Trivial Bundle, Path Lifting Theorem ensures that any path in the Base Space can be lifted to the Total Space but something stronger holds for Covering Spaces:

Theorem 6. *Let $\pi : \tilde{X} \to X$ be a covering space and x_0 be a point in X and $\tilde{x_0}$ a point in $\pi^{-1}(x_0)$. Suppose Z is a connected space and $f : Z \to X$ is a continuous map with $f(z_0) = x_0$. If there is a lift $\tilde{f} : Z \to \tilde{X}$ of f to $\tilde{X}$ with $\tilde{f}(z_0) = \tilde{x}_0$, then this lift is unique.*

Proof. Let there exist 2 continuous maps $\tilde{f}_1, \tilde{f}_2 : Z \to \tilde{X}$ that satisfy $\tilde{f}_1(z_0) = \tilde{f}_2(z_0) = \tilde{x}_0$ and $\pi \circ \tilde{f}_1 = \pi \circ \tilde{f}_2 = f$.

The connectedness of Z is the crucial fact for the proof.

Let $H = \{z \in Z : \tilde{f}_1(z) = \tilde{f}_2(z)\}$ and $K = Z - H = \{z \in Z : \tilde{f}_1(z) \neq \tilde{f}_2(z)\}$. We show that both H and K are open in Z. Then by the connectedness of Z either H or K is a null set. But we know that $z_0 \in H$. So K is empty and hence $\tilde{f}_1$ and $\tilde{f}_2$ will coincide everywhere and the proof follows.

Let $z_1 \in Z$ and let V be a locally trivializing neighbourhood (for Covering Spaces it is specifically called "evenly covered neighbourhood") of $f(z_1)$ in X. We separately consider the 2 mutually exclusively cases that $z_1 \in H$ or $Z_1 \in K$.

Let $z_1 \in H$. We shall try to show that z_1 is contained in an open-set inside H. But $\tilde{f}_1(z_1) = \tilde{f}_2(z_1)$ lies in some sheet S over V. Then $U = \tilde{f}_1^{-1}(S) \cap \tilde{f}_2^{-1}(S)$ is an open neighbourhood of $z_1 \in Z$. Both $\tilde{f}_1$ and $\tilde{f}_2$ map U into S and π is a homeomorphism on S so $\pi \circ \tilde{f}_1(z) = \pi \circ \tilde{f}_2(z) = f(z)$ for every $z \in U$ and hence it implies that $\tilde{f}_1(z) = \tilde{f}_2(z)$ for every $z \in U$.

Thus $z_1 \in U \subseteq H$ so that H is open.

Let $z_1 \in K$. We shall try to show that z_1 is contained in an open-set inside K. But $\tilde{f}_1(z_1) \neq \tilde{f}_2(z_1)$. But by definition $\pi \circ \tilde{f}_1(z_1) = \pi \circ \tilde{f}_2(z_1) = f(z_1)$. Hence $\tilde{f}_1(z_1)$ and $\tilde{f}_2(z_1)$ must lie in different sheets S_1 and S_2 over V. Then $W = \tilde{f}_1^{-1}(S_1) \cap \tilde{f}_2^{-1}(S_2)$ is an open neighbourhood of z_1 that $\tilde{f}_1$ and $\tilde{f}_2$ carry to different sheets over V. Since $S_1 \cap S_2 = \phi$, $\tilde{f}_1$ and $\tilde{f}_2$ disagree everywhere on W.

Thus $z_1 \in W \subseteq K$ and K is open.

Hence the proof. □

10.1. **Few Remarks.** The following consequences are almost trivially implied by the Unique Lifting Theorem:

- Let $\pi : \tilde{X} \to X$ be a covering space and x_0 be a point in X and $\tilde{x_0}$ be a point in $\pi^{-1}(x_0)$. Suppose $\alpha : [0,1] \to X$ be a path in X with $\alpha(0) = x_0$. Then there exists an unique lift $\tilde{\alpha} : [0,1] \to \tilde{X}$ of α to $\tilde{X}$ with $\tilde{\alpha}(0) = \tilde{x}_0$.

  The above follows from the just proved theorem by thinking of $[0,1]$ (which is connected) as Z.
- **Theorem 7.** *Let $\pi : \tilde{X} \to X$ be a covering space. Suppose $\phi_1, \phi_2 : \tilde{X} \to \tilde{X}$ are continuous maps for which $\pi \circ \phi_1 = \pi \circ \phi_2 = \pi$. If there exists a $p \in \tilde{X}$ for which $\phi_1(p) = \phi_2(p)$, then $\phi_1 = \phi_2$.*

  Proof. By definition of being a covering space $\tilde{X}$ is a connected space and π is a continuous map. So in the Unique Lifting Theorem Z can be $\tilde{X}$ and f can be π. z_0 can be p and x_0 can be $\pi \circ \phi_1(p) = \pi \circ \phi_2(p) = \pi(p)$. So the Unique Lifting Theorem guarantees the uniqueness of the lift of $\pi : \tilde{X} \to X$ to $\tilde{\pi} : \tilde{X} \to \tilde{X}$. This $\tilde{\pi}$ is the candidate for ϕ_1 or ϕ_2. Hence proved. □

  Such ϕ are called **Deck Tranformations** and the set of such Deck Transformations form a group and is called the **Galois Group** of the covering space and this is a subgroup of the group of automorphisms of the covering space and these are of extreme importance in various areas like in Homotopy Theory.
- In the above two theorems one was working on Locally Trivial Bundles. But some of the results also hold if one weakens the structure to just a Local Homeomorphism. If the pojection map from the total space to the base space is a local homeomorhism then, unlike locally trivial bundles, here one cannot guarantee existence of lifts of paths in the base-space. But if there exists lifts then the following holds:

  Theorem 8. *Let $p : X \to Y$ be a local homomorphism and Z be a connected Hausdorff space. Let $f : Z \to Y$ be a contnuous map and suppose f_1 and f_2 are two liftings of f. Then if there exists a point $z_0 \in Z$ such that $f_1(z_0) = f_2(z_0)$ then we have $f_1 = f_2$.*

  The above is proved in exactly the same way as the Unique Lifting Theorem and the connectedness of Z is crucial to the proof.
- One can show that for a local homeomorphism if two paths in the base space with the same starting point are homotopic to each other then the end points of their lifts are same (provided both the paths can be lifted). As a special case this holds for Locally Trivial Bundles (Vector Bundles and Principal Bundles) and for Covering Spaces. This is the essence of the **Monodromy Theorem** and it has elegant intepretations in the framework of Sheaf Theory.

  The above is very crucial in the construction of Principal Bundles over a manifold with its first homotopy group being the strucure group.

11. Final Concluding Remarks

The Path Lifting Theorem and Unique Lifting Theorem has wide spread ramifications especially in understanding the Homotopy properties of fibre bundles and how sometimes to understand the homotopy property of a space we need to imagine a fiber bundle over it and then start investigating.

But more excitingly these ideas of lifting of paths become the seeds of deep ideas in physics through a mathematical reformulation of Gauge Field Theory.

At the end of it all its all about lifts of maps and specifying enough conditions to ensure unique lifts (if it exits!).

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