Search for the QCD Critical Point: Hints from Lattice

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Introduction

Lattice QCD Results

Searching Experimentally

Summary
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♠ QCD Critical Point in $T-\mu_B$ plane.

- Search for its location using *ab initio* methods
- Search for it in the experiments RHIC, FAIR,...

From Rajagopal-Wilczek Review
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- Search for its location using \textit{ab initio} methods
- Search for it in the experiments RHIC, FAIR,...
- What hints can Lattice QCD investigations provide?

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The $\mu \neq 0$ problem: Quark Type

- Mostly staggered quarks used in these simulations. Broken flavour and spin symmetry on lattice $\implies N_f = 2$ simulations may be fine in $a \to 0$ limit but 3 or 2 +1 problematic.
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- Unfortunately BW-prescription breaks chiral symmetry! (Banerjee, Gavai & Sharma PRD 2008; PoS (Lattice 2008); PRD 2009) Furthermore, anomaly for it depends on $\mu$ unlike in continuum QCD (Gavai & Sharma PRD 2010).

- Desperately needed: Formalism with Continuum-like (flavour & spin) symmetries for quarks at nonzero $\mu$ and $T$. 
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How Do We Do This Expansion?

Canonical definitions yield various number densities and susceptibilities:

\[ n_i = \frac{T}{V} \frac{\partial \ln Z}{\partial \mu_i} \quad \text{and} \quad \chi_{ij} = \frac{T}{V} \frac{\partial^2 \ln Z}{\partial \mu_i \partial \mu_j}. \]

These are also useful by themselves both theoretically and for Heavy Ion Physics (Flavour correlations, \( \lambda_s \ldots \))

Denoting higher order susceptibilities by \( \chi_{n_u,n_d} \), the pressure \( P \) has the expansion in \( \mu \):

\[ \frac{\Delta P}{T^4} \equiv \frac{P(\mu, T)}{T^4} - \frac{P(0, T)}{T^4} = \sum_{n_u,n_d} \chi_{n_u,n_d} \frac{1}{n_u!} \left( \frac{\mu_u}{T} \right)^{n_u} \frac{1}{n_d!} \left( \frac{\mu_d}{T} \right)^{n_d} \quad (1) \]
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\]
• We construct the series for baryonic susceptibility from this expansion. Its radius of convergence gives the nearest critical point.

• Successive estimates for the radius of convergence obtained from these using
\[
\sqrt{\frac{n(n+1)\chi_B^{(n+1)}}{\chi_B^{(n+3)}T^2}} \quad \text{or} \quad \left( n!\frac{\chi_B^{(2)}}{\chi_B^{(n+2)}T^n} \right)^{1/n}
\]
. We use both these definitions.

• All coefficients of the series must be POSITIVE for the critical point to be at real \( \mu \), and thus physical.

• We (Gavai-Gupta ’05, ’09) use up to 8\textsuperscript{th} order. Need 20 inversions of \((D + m)\) on \(\sim 500\) vectors for a single measurement.

• Ideas to extend to higher orders are emerging (Gavai-Sharma PRD 2010) which save up to 60 % computer time: 10th & 12th order may be possible.
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Lattice QCD Results

- Staggered fermions with $N_f = 2$ of $m/T_c = 0.1$; R-algorithm used.
- $m_\pi = 230$ MeV.
- Earlier Lattice: $4 \times N_s^3$, $N_s = 8, 10, 12, 16, 24$ (Gavai-Gupta, PRD 2005)
- Finer Lattice: $6 \times N_s^3$, $N_s = 12, 18, 24$ (Gavai-Gupta, PRD 2009). We determined $\beta_c$. Our result ($\beta_c = 5.425(5)$) well bracketed by MILC for $m/T_c = 0.075$ and 0.15.
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- Our Simulations made for $0.89 \leq T/T_c \leq 1.92$. Typical stat. 50-200 in autocorrelation units.
- The same configurations being used for our new proposal of $\mu N$ term.
Using our proposed $\mu N$ term (Gavai-Sharma PRD 2010) to evaluate the baryon susceptibility at $\mu = 0$,
Preliminary Results with $\mu N$-idea

♠ Using our proposed $\mu N$ term (Gavai-Sharma PRD 2010) to evaluate the baryon susceptibility at $\mu = 0$,

♠ The estimates for radius of convergence are comparable as well.
♥ $\chi_8 < 0$ in both cases at $1.92 \ T_c$. 

Extreme QCD 2010, 461. WE-Heraeus Seminar, Physikzentrum, Bad Honnef, June 22, 2010
$\mu/(3T)$

$T/T_c = 0.99$

Diagram shows a plot of $\mu/(3T)$ versus $n$ with a specific point indicated for $T/T_c = 0.99$. The data points are marked with error bars.
T/Tc = 0.99
T/Tc = 0.97
\begin{itemize}
  \item \( \frac{T^E}{T_c} = 0.94 \pm 0.01 \), and \( \frac{\mu_B^E}{T^E} = 1.8 \pm 0.1 \) for finer lattice: Our earlier coarser lattice result was \( \mu_B^E/T^E = 1.3 \pm 0.3 \). Infinite volume result: ↓ to 1.1(1)
  \item Critical point at \( \mu_B/T \sim 1 - 2 \).
\end{itemize}
Cross Check on $\mu^E/T^E$

♠ Use the series directly to construct $\chi_B$ for nonzero $\mu \longrightarrow$ smooth curves with no signs of criticality.

![Graph showing $\chi_B/T^2$ vs $\mu/T$ for different values of $n$.]
Cross Check on $\mu^E/T^E$

♠ Use the series directly to construct $\chi_B$ for nonzero $\mu \rightarrow$ smooth curves with no signs of criticality.

♥ Use Padé approximants for the series to estimate the radius of convergence.

♥ Consistent Window with our other estimates.
Searching Experimentally

- Hadron yields well described using Thermodynamical Models, leading to a freezeout curve in the $T-\mu_B$ plane. (Andronic, Braun-Munzinger & Stachel, PLB 2009; Oeschler, Cleymans, Redlich & Wheaton, 2009)
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Plotting these results in the \( T-\mu_B \) plane, one has the freezeout curve, which was shown to correspond the \( \langle E \rangle / \langle N \rangle \simeq 1 \). (Cleymans and Redlich, PRL 1998)

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![Graph showing freezeout curve](image)

(From Braun-Munzinger, Redlich and Stachel nucl-th/0304013)

• Key point: Freeze-out curve, based solely on data on hadron yields, gives the $(T, \mu)$ accessible in heavy-ion experiments.
coexistence curve
Freezeout curve

Coexistence curve

$T/T_c$

$\mu_B/T$
Use the freezeout curve computed from hadron abundances to relate $(T, \mu_B)$ to $\sqrt{s}$ and employ lattice QCD predictions along it. (Gavai-Gupta, TIFR/TH/10-01, arXiv 1001.3796)
• Use the freezeout curve computed from hadron abundances to relate \((T, \mu_B)\) to \(\sqrt{s}\) and employ lattice QCD predictions along it. (Gavai-Gupta, TIFR/TH/10-01, arXiv 1001.3796)

• Define \(m_1 = \frac{T\chi^{(3)}(T,\mu_B)}{\chi^{(2)}(T,\mu_B)}\), \(m_3 = \frac{T\chi^{(4)}(T,\mu_B)}{\chi^{(3)}(T,\mu_B)}\), and \(m_2 = m_1 m_3\) (Gupta, arXiv : 0909.4630) and use the Padè method to construct them.
Near the critical point, $\chi_B \sim |\mu - \mu_E|^\delta$. Thus the ratios of successive NLS, $m_i$, should diverge in the critical region as well.

Spatial Volume cancels out in these ratios $\implies$ Suitable for experiments who can use their favourite proxy for it.
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• Defining $z = \mu_B/T$, and denoting by $r_{ij}$ the estimate for radius of convergence using $\chi_i, \chi_j$, one has

$$m_1 = \frac{2z}{r_{24}^2} \left[ 1 + \left( \frac{2r_{24}^2}{r_{46}^2} - 1 \right) z^2 + \left( \frac{3r_{24}^2}{r_{46}^2r_{68}^2} - \frac{3r_{24}^2}{r_{46}^2} + 1 \right) z^4 + O(z^6) \right].$$

• Similar series expressions for $m_2$ and $m_3$. Resum these by Padè ansatz:

$$m_1 = z P_1^1(z^2; a, b), \quad m_3 = \frac{1}{z} P_1^1(z^2; a', b')$$

.
• Use $T_c(\mu = 0) = 175$ MeV; for $N_t = 4$ (boxes) and 6 (circles).
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- Our estimated critical point suggests a peak-like structure in all $m_i$ which would be accessible to the low energy scan of RHIC BNL!!

- Smooth & monotonic behaviour without the critical point.
Note, however, that even in the smooth region, an experimental comparison is exciting: Non-perturbative test of QCD in hot and dense environment.
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• Proton number fluctuations (Hatta-Stephenov, PRL 2003)

• Directly linked to the baryonic susceptibility which ought to diverge at the critical point. Since diverging $\xi$ is linked to $\sigma$ mode, which cannot mix with any isospin modes, expect $\chi_I$ to be regular.
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• Leads to a ratio $\chi_Q:\chi_I:\chi_B = 1:0:4$

• Assuming protons, neutrons, pions to dominate, both $\chi_Q$ and $\chi_B$ can be shown to be proton number fluctuations only.

• STAR has recently used this idea and constructed the ratios $m_1$ and $m_2$ from net proton distributions: (arXiv: 1004.4959).
Aggarwal et al., STAR Collaboration, arXiv: 1004.4959
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- Reasonable agreement with our lattice results. Where is the critical point?
Private communication from STAR Collaboration

### (a) Au+Au Collisions

**Most Central**

0.4<p_T<0.8 (GeV/c), |y|<0.5

#### Lattice QCD

- △ N_c=4
- ○ N_c=6

### (b) Net-proton

**\(\kappa \sigma^2\)**

<table>
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<th>(\sqrt{s_{NN}}) (GeV)</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>20</th>
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<td>(\kappa \sigma^2)</td>
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Summary

- Phase diagram in $T - \mu$ has begun to emerge: Different methods, $\sim$ similar qualitative picture. Critical Point at $\mu_B/T \sim 1 - 2$. 


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- STAR results appear to agree with our Lattice QCD predictions.
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So far no signs of a critical point in the experimental results at CERN. Will RHIC energy scan deliver it for us? and/or Will it be FAIR?
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- Ease of taking continuum and thermodynamic limit.
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We study volume dependence at several \( T \) to i) bracket the critical region and then to ii) track its change as a function of volume.
For $N_f = 3$, they find $\frac{m_c(\mu)}{m_c(0)} = 1 - 3.3(3) \left( \frac{\mu}{\pi T_c} \right)^2 - 47(20) \left( \frac{\mu}{\pi T_c} \right)^4$, i.e., $m_c$ shrinks with $\mu$. 
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Problems: i) Positive coefficient for finer lattice (Philipsen, CPOD 2009), ii) Known examples where shapes are different in real/imaginary $\mu$.