Quark Gluon Plasma

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- Introduction
- QGP in Bulk
- ${\rm J}/\psi$ suppression
- Jet Quenching
- QCD Phase Diagram
- Summary





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- Lattice QCD only well-understood, viable tool for this: 'Iconic Results' Wilczek.
- It predicts a transition to Quark-Gluon Plasma (QGP), and QGP properties.

♠ Quest for Quark-Gluon Plasma : Heavy Ion Collisions at SPS, RHIC and LHC.



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• Completely parameter-free : Λ_{QCD} and quark masses from hadron spectrum.

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• Other quantities, notably strangeness enhancement in Heavy Ion Physics, the Wróblewski Parameter λ_s (RVG & Sourendu Gupta PR D 2002) have also been predicted by lattice QCD.

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- continuum limit, lighter quarks, 2+1 flavours \rightsquigarrow transition temperature $T_c = 192(7)(4)$ MeV (Bielefeld-RBC hep-lat/0608013).

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- An interesting theoretical issue Conformal Invariance and AdS/CFT predictions.
- Lot of activity in Model Building to explain Lattice QCD results: Quasi-particle models, Hadron Resonance Gas, Quarkonia from Lattice $Q\bar{Q}$ potential, sQGP and coloured states...

QGP in Bulk : EoS, Speed of Sound...

• Recent results for EoS : $N_t=6$, Smaller quark masses.



Bernard et al., MILC hep-lat/0509053;

Aoki et al., hep-lat/0510084.

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- New method to obtain these differentially without getting negative pressure. Introduced an improved operator than used in earlier Bielefeld studies. (RVG, S. Gupta and S. Mukherjee, hep-lat/0506015)
- Using lattices with 8, 10, and 12 temporal sites $(38^3 \times 12 \text{ and } 38^4 \text{ lattices})$ and with statistics of 0.5-1 million iterations, ϵ , P, s, C_s^2 and C_v obtained in continuum.



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- $\blacklozenge C_v \sim 4\epsilon$ for $2T_c$ but No Ideal Gas limit.
- Specific heat \iff fluctuations in p_T ?
- \blacklozenge C_s^2 closer to Ideal Gas limit; Any structure near T_c ??

• Entropy agrees with strong coupling SYM prediction (Gubser, Klebanov & Tseytlin, NPB '98, 202) for $T = 2 - 3T_c$ but fails at lower T, as do various weak coupling schemes : $\frac{s}{s_0} = f(g^2N_c)$, where $f(x) = \frac{3}{4} + \frac{45}{32}\zeta(3)x^{-3/2} + \cdots$ and $s_0 = \frac{2}{3}\pi^2N_c^2T^3$. • Entropy agrees with strong coupling SYM prediction (Gubser, Klebanov & Tseytlin, NPB '98, 202) for $T = 2 - 3T_c$ but fails at lower T, as do various weak coupling schemes : $\frac{s}{s_0} = f(g^2N_c)$, where $f(x) = \frac{3}{4} + \frac{45}{32}\zeta(3)x^{-3/2} + \cdots$ and $s_0 = \frac{2}{3}\pi^2N_c^2T^3$.



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v₂ at Low p_T Region



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mm





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$$\Gamma_s = \frac{\frac{4}{3}\eta}{sT} , \qquad (1)$$

where η is Shear Viscosity and s is entropy density; $\tau = \sqrt{t^2 - z^2}$ is the time scale of expansion.



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Perturbation theory \Rightarrow Large η/s Small $\eta/s \longrightarrow$ Strongly Coupled Liquid.



Nakamura and Sakai, PRL 94 (2005).



- Kubo's Linear Theory 1 Coefficients in equilibrium functions.
- Response Transport terms of correlation

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- Obtain Energy-Momentum Correlation functions on Lattice (at discrete Matsubara frequencies).
- Continue them to get Retarded ones → Shear, Bulk Viscosities.



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- Kubo's Linear Response Theory : Transport Coefficients in terms of equilibrium correlation functions.
- Obtain Energy-Momentum Correlation functions on Lattice (at discrete Matsubara frequencies).
- Continue them to get Retarded ones → Shear, Bulk Viscosities.
- Larger lattices and inclusion of dynamical quarks in future.

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- Fluctuation-Dissipation Theorem \longrightarrow Production of Strange quark-antiquark pair \sim imaginary part of generalized strange quark susceptibility.
- Kramers Krönig relation can be used to relate it to the real part of the susceptibility, which we obtain from lattice QCD simulations.
- Finally, make a relaxation time approximation $(\omega \tau \gg 1) \rightsquigarrow$ ratio of real parts is the same as the ratio of imaginary parts.

We use $m/T_c = 0.03$ for u, d and $m/T_c = 1$ for s quark; At each T, ratio of χ_s and $\chi_{ud} \rightarrow \lambda_s(T)$.

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Baryon Number(Charge)–Strangeness correlation : $C_{(BS)/S}$ ($C_{(QS)/S}$) (Koch, Majumdar and Randurp, PRL 95 (2005); RVG & Sourendu Gupta, PR D 2006; S. Mukherjee, hep-lat/0606018); u-d Correlation.

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Anomalous J/ψ Suppression : CERN NA50 results

♠ Matsui-Satz idea — J/ψ suppression as a signal of QGP. ♠ Deconfinement \rightsquigarrow Screening of coloured quarks, which cannot bind.

Anomalous J/ψ Suppression : CERN NA50 results

$\sigma_{abs}(J/\psi) = 4.18 \pm 0.35 \text{ mb}$ $\sigma_{abs}(\psi') = 7.60 \pm 1.12 \text{ mb}$ Measured / Expected 1.4 1.4 1.2 1.4 S-U D p-A 0.8 0.6 0.4 02 0 2 8 10 4 6 L (fm)

Expected = Glauber absorption model

- **S-U** and **peripheral Pb-Pb (J/ψ)/DY** results follow the absorption curve extrapolated from p-A measurements.
- **Pb-Pb central** collisions show an **anomalous** $(J/\psi)/DY$ suppression with respect to p-A behaviour.
- **ψ'/DY** behaviour is the same in S-U and Pb-Pb interactions and not compatible with the one observed in p-A collisions.
- ψ **anomalous suppression** sets in earlier than the J/ ψ one.

System-Size Dependence



Models that were successful in describing SPS data

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- too much suppression -

R. V. Gavai Top



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- A critical assessment of the original theoretical argument: Made feasible by the recognition of MEM technique as a tool to extract spectral functions from the temporal correlators computed on the Euclidean lattice.
- Caution : nonzero temperature obtained by making temporal lattices shorter : $48^3 \times 12$ to $64^3 \times 24$ Lattices used. (S. Datta et al., Phys. Rev. D 69, 094507 (2004).)



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No Significant Effect of inclusion of dynamical fermions ?

Jet Quenching



Jet Quenching



- Rare, Highly Energetic Scatterings produce jets of particles : $g + g \rightarrow g + g$.
- Quark-Gluon Plasma, any medium in general, interacts with a jet, causing it to lose energy Jet Quenching.

 On-Off test possible – Compare Collisions of Heavy-Heavy nuclei with Light-Heavy or Light-Light. • On-Off test possible – Compare Collisions of Heavy-Heavy nuclei with Light-Heavy or Light-Light.



What is new: di-hadron correlations at higher DT



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Expected QCD Phase Diagram and Lattice Approaches to unravel it.


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- Taylor Expansion (C. Allton et al., PR D66 (2002)
 074507 & D68 (2003) 014507; R.V. Gavai and S. Gupta, PR
 D68 (2003) 034506).

Critical Point Estimate



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Critical Point Estimate



A Radii of convergence as a function of the order of expansion at $T = 0.95T_c$ on $N_s = 8$ (circles) and 24 (boxes). Left panel for ρ_n and right one for r_n .

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• Extrapolation in $n \rightsquigarrow \mu^E/T^E = 1.1 \pm 0.2$ at $T^E = 0.95T_c$. Finite volume shift consistent with Ising Universality class.

Summary

- Lattice QCD **predicts** transition to Quark-Gluon Plasma and several of its properties, T_c , EoS, λ_s , η ...
- RHIC data exhibits tell-tale signs of QGP : Flow, Jet Quenching, J/ψ BUT agreement with theory still not very quantitative, e.g., v_2 .

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Quarkonia moving in the Heat Bath

 \blacklozenge Should see more energetic gluons. More Dissociation at the same T as momentum of J/ψ increases ? Datta et al. SEWM 2004, PANIC 2005.

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• Both J/ψ and η_c do show this trend. • The effect is significant at both 0.75 and $1.1T_c$.

$m_{ ho}/T_c$	$m_{\pi}/m_{ ho}$	$m_N/m_{ ho}$	$N_s m_{\pi}$	flavours	T^E/T_c	μ_B^E/T^E
5.372 (5)	0.185 (2)		1.9–3.0	2+1	0.99 (2)	2.2 (2)
5.12 (8)	0.307 (6)		3.1–3.9	2 + 1	0.93 (3)	4.5 (2)
5.4 (2)	0.31(1)	1.8 (2)	3.3–10.0	2	0.95 (2)	1.1 (2)
5.4 (2)	0.31(1)	1.8 (2)	3.3	2		
5.5 (1)	0.70(1)		15.4	2		

Table 1: Summary of critical end point estimates— the lattice spacing is a = 1/4T. N_s is the spatial size of the lattice and $N_s m_{\pi}$ is the size in units of the pion Compton wavelength, evaluated for $T = \mu = 0$. The ratio m_{π}/m_K sets the scale of the strange quark mass.

Results are sequentially from Fodor-Katz '04, Fodor-Katz '02, Gavai-Gupta, de Forcrand- Philipsen and Bielefeld-Swansea.