Statistics of Andreev conductance in superconductor-metal junctions

Random matrix theory for asymmetric large-deviation tails

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Reference:
K. Damle, S.N. Majumdar, V. Tripathi, P. Vivo, PRL **107**, 177206 (2011).

Conductance a la Landauer

Two-terminal measurement

Landauer Formula

$$\mathbb{S} = \begin{pmatrix} \mathbf{r}_{11}(\epsilon) & \mathbf{t}_{12}(\epsilon) \\ \mathbf{t}_{21}(\epsilon) & \mathbf{r}_{22}(\epsilon) \end{pmatrix}$$

$$G \equiv \frac{I}{V_1 - V_2} = \frac{2e^2}{h} \mathbf{Tr} \left(\mathbf{t}_{21}(\epsilon_F) \mathbf{t}_{21}^{\dagger}(\epsilon_F) \right)$$

in the $k_B T \to 0$ limit

Landauer (57); Fisher & Lee (81); Imry (86); Buttiker (86)

Generalizations

Finite temperature

$$G \equiv \frac{I}{V_1 - V_2} = \frac{2e^2}{h} \int d\epsilon (-\frac{df}{d\epsilon}) \mathbf{Tr} \left(\mathbf{t}_{21}(\epsilon) \mathbf{t}_{21}^{\dagger}(\epsilon) \right)$$

Shot noise at finite bias V

$$P(\omega) = 2 \int dt e^{i\omega t} \langle \Delta I(t_0 + t) \Delta I(t_0) \rangle$$

$$P(\omega \to 0) = 2eV \frac{e^2}{h} \mathbf{Tr} \left(\mathbf{t}_{21} \mathbf{t}_{21}^{\dagger} (\mathbf{1} - \mathbf{t}_{21} \mathbf{t}_{21}^{\dagger}) \right)$$

in the $k_B T \to 0$ limit

Random matrix theory approach

Idea

- Realistic description of disordered/chaotic dynamics in device beyond reach
- Either simplify $\rightarrow$
  Point-disorder models—impurity-averaged perturbation theory
- Or factor in our "ignorance" $\rightarrow$
  random matrix ensemble for $\mathbb{S}$

Imry (86); Muttalib, Pichard, & Stone (87); Baranger & Mello (94); Jalabert, Pichard, & Beenakker (94)

Ingredients

- Appropriate ensemble for $\mathbb{S}$: "Uniform" distribution over "all" unitary ($\mathbb{S}^\dagger\mathbb{S} = \mathbf{1}$) matrices.
- Corresponding statistics for $\mathbf{t}_{21}\mathbf{t}_{21}^\dagger$

Meanings of “uniform” and “all”

- If system has no time-reversal symmetry: $\mathbb{S}$ is “equally likely” to be *any* unitary matrix—Haar measure
- If system has time reversal symmetry and spin-rotation invariance: $\mathbb{S}$ must be a orthogonal matrix—equally likely to be *any orthogonal* matrix
- If system has time reversal invariance but no spin-rotation invariance: More complicated.

Distribution of eigenvalues of $\mathbf{t}_{21}\mathbf{t}_{21}^{\dagger} \equiv \mathrm{diag}(T_1, T_2 \ldots T_{N_c})$

Our system: No magnetic field, no spin-orbit scattering
T-symmetry, spin symmetry
$\rightarrow$
The “Jacobi” Orthogonal random matrix ensemble

$$\mathcal{P}_{\mathbf{T}}(\{T_n\}) = A_{N_c} \prod_{n<m} |T_n - T_m| \prod_n T_n^{-1/2},$$

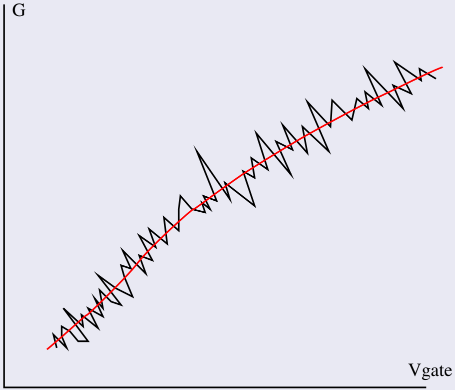
with A_{N_c} ensuring normalization.

- Small N_c $\rightarrow$
  Can compute G_{av} $(\Delta G)^2_{av}$, even *full* $P(G)$ by direct integration

Beenakker, RMP review (97)

Interpretation of $P(G)$

Experimental trace



Theorist's caricature

- Gate voltage creates "ensemble" of devices
- Subtract out drift and histogram $\rightarrow P(G)$

Consequences

Universal conductance fluctuations

- $G = \sum_i T_i$ with T_i strongly correlated
- Consequence: $\langle(\Delta G)^2\rangle$ independent of size (N_c) in the large N_c limit.
- *Different* from central-limit considerations

Imry (86), Muttalib, Pichard, & Stone (87)

Our focus: Andreev Conductance (and spin-conductance)

Recent progress

- Full distribution of $P(G)$ beyond central Gaussian regime around G_{av} for N_c large. Vivo, Majumdar, & Bohigas (2008)

Our interest: Generalize to Andreev conductance & spin-conductance
Spin-conductance: Hard!
Today: Andreev conductance

Andreev conductance

Set-up

- Normal metal - superconductor junction
- Two-terminal conductance affected by Andreev processes: Electron reflecting as hole and injecting cooper pair into superconductor

Formalism

$$G_{\mathrm{NS}} = 2 \sum_{n=1}^{N_c} \left(\frac{T_n}{2 - T_n} \right)^2$$

in units of $G_0 = 2e^2/h$

Valid *only* for $B = 0$ linear response (Beenakker 92)

$P(G_{NS})$: Formal expression

$$\mathcal{P}(G_{\rm NS}, N_c) =$$

$$= \int_{[0,1]^{N_c}} \prod_i dT_i \delta\left(g_{\rm NS} N_c - 2\sum_{n=1}^{N_c} \frac{T_n^2}{(2-T_n)^2} \right) \mathcal{P}_{\mathbf{T}}\left(\{T_n\}\right).$$

With $\xi_n = T_n/(2-T_n)$:
$\mathcal{P}(G_{\rm NS}, N_c) =$

$$= \frac{N_c}{2} \int \frac{d\kappa}{2\pi} \int_{[0,1]^{N_c}} \prod_i d\xi_i \; e^{\mathrm{i} N_c^2 \kappa \left(\frac{1}{N_c} \sum_{n=1}^{N_c} \xi_n^2 - \frac{g_{\rm NS}}{2} \right)} \mathcal{P}_\xi\left(\{\xi_n\}\right),$$

where

$$\mathcal{P}_\xi\left(\{\xi_n\}\right) = \tilde{A}_{N_c} \prod_{n<m} |\xi_n - \xi_m| \prod_n \frac{\xi_n^{-1/2}}{(1+\xi_n)^{N_c+3/2}}$$

Rewrite to seek saddle-point at large-N_c:

$$\mathcal{P}_\xi\left(\{\xi_n\}\right) \propto e^{-N_c^2 \mathcal{F}(\{\xi_n\}) + \mathcal{O}(N_c)} \tag{1}$$

where:

$$\mathcal{F}(\{\xi_n\}) := \frac{1}{N_c}\sum_{i=1}^{N_c} \ln(1+\xi) - \frac{1}{2N_c^2}\sum_{j\neq k} \ln|\xi_j - \xi_k| \tag{2}$$

Combining everything together we get:

$$\mathcal{P}(G_{\mathrm{NS}}) \sim \frac{N_c}{2}\int \frac{d\kappa}{2\pi}\int_{[0,1]^{N_c}} \prod_i d\xi_i \, e^{-N_c^2\left[-\mathrm{i}\kappa\left(\frac{1}{N_c}\sum_{n=1}^{N_c}\xi_n^2 - \frac{g_{\mathrm{NS}}}{2}\right) + \mathcal{F}(\{\xi_n\})\right]}, \tag{3}$$

Continuum Coulomb gas formulation at large N_c:

Define density $\rho(\xi) = \frac{1}{N_c}\sum_{n=1}^{N_c} \delta(\xi - \xi_n)$.

$$\mathcal{P}(G_{\rm NS}, N_c) = \mathcal{A}_{N_c} \int d\kappa \int d\chi \int \mathcal{D}\rho \exp\left(-N_c^2 \mathcal{S}[\rho]\right),$$

with

$$\mathcal{S}[\rho] := -\mathrm{i}\chi\left(\int d\xi\rho(\xi) - 1\right) - \mathrm{i}\kappa\left(\int d\xi\rho(\xi)\xi^2 - \frac{g_{\rm NS}}{2}\right) + \\ + \int d\xi\rho(\xi)\ln(1+\xi) - \frac{1}{2}\int\int d\xi d\xi'\rho(\xi)\rho(\xi')\ln|\xi - \xi'| \quad (4)$$

and $\mathcal{A}_{N_c} \sim \exp(3N_c^2(\ln 2)/2)$.
Integrals over χ and κ enforce $\int d\xi\rho(\xi) = 1$ (normalization of the density field) and $\int d\xi\rho(\xi)\xi^2 = \frac{g_{\rm NS}}{2}$.

Saddle-point at large N_c

$$\frac{\partial}{\partial \kappa}\mathcal{S}[\rho] = -\mathrm{i}\left(\int d\xi \rho(\xi)\xi^2 - \frac{g_{\mathrm{NS}}}{2}\right) = 0 \tag{5}$$

$$\frac{\partial}{\partial \chi}\mathcal{S}[\rho] = -\mathrm{i}\left(\int d\xi \rho(\xi) - 1\right) = 0 \tag{6}$$

$$\frac{\delta}{\delta \rho}\mathcal{S}[\rho] = -\mathrm{i}\kappa\xi^2 - \mathrm{i}\chi + \ln(1+\xi) - \int d\xi' \rho(\xi') \ln|\xi - \xi'| = 0 \tag{7}$$

So $\kappa = \mathrm{i}C_1$ and $\chi = \mathrm{i}C_0$ with C_0 and C_1 real

Saddle-point equations for the large N_c limit

Key point:

$$\ln(1+\xi) + C_0 + C_1\xi^2 = \int \rho^\star(\xi^{'}) \ln|\xi - \xi^{'}| d\xi^{'}$$

only for ξ in the support of saddle-point density $\rho^\star$.
Differentiating with respect to ξ:

$$2C_1\xi + \frac{1}{1+\xi} = \mathrm{Pr} \int \frac{\rho^\star(\xi^{'})}{\xi - \xi^{'}} d\xi^{'}$$

for all ξ in the support of $\rho^\star$ (Pr stands for Cauchy's principal part.)

Physical picture:

- Charged particles with log repulsion and external potential $C_1\xi^2 + \ln(1+\xi)$
- Logarithmic repulsion tries to spread density out uniformly
- Large positive C_1 tries to pile up density at left edge (near $\xi = 0$)
- Large negative C_1 tries to pile up density at right edge (near $\xi = 1$)
- When $|C_1|$ small, situation unclear. Log interaction tries to spread out density. What does log potential do?
  Later...

Connection of C_1 to g_{NS}

- Since $2\int d\xi\xi^2\rho^\star(\xi) = g_{\mathrm{NS}}$: $g_{\mathrm{NS}} \to 0$ corresponds to large positive value of C_1
- $g_{\mathrm{NS}} \to 2$ corresponds to large negative C_1.
- g_{NS} in the "middle": $|C_1|$ small.

Method of solution

Find $\rho^\star$ s.t.

$$V'(\xi) = \text{Pr} \int \frac{\rho^\star(\xi')}{\xi - \xi'} d\xi'$$

for all ξ in the support of $\rho^\star$

Tricomi:

If $\rho^\star$ has support on a single interval (L_1, L_2), then

$$\rho^\star = -\frac{1}{\pi^2 \sqrt{(L_2 - \xi)(\xi - L_1)}} \times \left(\text{Pr} \int_{L_1}^{L_2} d\xi' \frac{\sqrt{(L_2 - \xi')(\xi' - L_1)}}{\xi - \xi'} V'(\xi') + \text{const.} \right)$$

Small $g_{\rm NS}$ (large positive C_1)

$$\rho^\star = -\frac{1}{\pi^2\sqrt{(L_2-\xi)\xi}} \times \left(\Pr \int_0^{L_2} d\xi' \frac{\sqrt{(L_2-\xi')\xi'}}{\xi-\xi'} (2C_1\xi' + \frac{1}{1+\xi'}) + \mathrm{const.} \right)$$

const determined by
$\rho^\star(L_2) = 0$.
C_1 and L_2 determined by
$\int d\xi \rho^\star(\xi) = 1$ and $\int d\xi \rho^\star(\xi)\xi^2 = \frac{g_{\rm NS}}{2}$

Form of $\rho^\star$ for small g_{NS}

$$\rho_I^\star(\xi) = \frac{\sqrt{L_1 - \xi}}{\pi\sqrt{\xi}} \left(\frac{1}{(\xi+1)\sqrt{L_1+1}} + C_1(L_1 + 2\xi) \right),$$

where $C_1 = \frac{4}{3L_1^2\sqrt{L_1+1}}$ and $1 + \frac{5L_1^2 - 8L_1 - 16}{16\sqrt{L_1+1}} = g_{\mathrm{NS}}/2$

Valid until g_{NS} reaches $g_1 = 2 - 19/8\sqrt{2} = 0.320621\ldots$

(L_1 hits 1 at $g_{\mathrm{NS}} = g_1$)

$g_{\rm NS}$ close to 2 (large negative C_1:

$$\rho^\star = -\frac{1}{\pi^2\sqrt{(1-\xi)(\xi-L_1)}} \times \left(\Pr\int_{L_1}^{1} d\xi' \frac{\sqrt{(1-\xi')(\xi'-L_1)}}{\xi-\xi'}(2C_1\xi' + \frac{1}{1+\xi'}) + \mathrm{const.} \right)$$

with const, L_1 and C_1 determined by demanding that
$\rho^\star(L_1) = 0$.
$\int d\xi \rho^\star(\xi) = 1$ and
$\int d\xi \rho^\star(\xi)\xi^2 = \frac{g_{\rm NS}}{2}$

Form of $\rho^\star$ for g_{NS} near 2

$$\rho^\star_{IV}(\xi) = \frac{\sqrt{2}}{\pi}\frac{\sqrt{\xi - L_4}}{\sqrt{1+L_4}}\frac{1}{\sqrt{1-\xi}}\times$$
$$\times\left(\frac{4(2\xi + L_4 - 1)}{(1-L_4)(1+3L_4)} - \frac{1}{1+\xi}\right),$$

where L_4 is determined by

$$\frac{\sqrt{2}(1-L_4)(1-18L_4-15L_4^2)}{16\sqrt{1+L_4}(1+3L_4)} = \frac{g_{\mathrm{NS}}}{2} - 1.$$

valid for $g_{NS} \geq g_3 \equiv 1.64939\ldots$
(no solution for L_4 for $g_{NS} < g_3$)

g_{NS} just greater than g_1?

$\rho^\star$ has support on $[0, 1]$:

$$\rho^\star = -\frac{1}{\pi^2\sqrt{(1-\xi)\xi}} \times \left(\Pr \int_0^1 d\xi' \frac{\sqrt{(1-\xi')\xi'}}{\xi - \xi'} (2C_1\xi' + \frac{1}{1+\xi'}) + \text{const.} \right)$$

with const and C_1 determined by
$\int d\xi \rho^\star(\xi) = 1$ and
$\int d\xi \rho^\star(\xi)\xi^2 = \frac{g_{\text{NS}}}{2}$

Form of $\rho^\star$ in this regime:

$$\rho^\star_{II}(\xi) = \frac{1}{\pi\sqrt{\xi(1-\xi)}}\left(\frac{\sqrt{2}}{\xi+1} + \frac{C_1}{4}(1+4\xi-8\xi^2)\right),$$

with $C_1 = \frac{32}{9}(2-\sqrt{2}-g_{\mathrm{NS}})$.

Intermediate asymptotics:

For $g_{\rm NS} > g_2 \equiv (968 - 499\sqrt{2} + 102\sqrt{17})/484 = 1.41088\ldots$, $\rho^\star_{II}$ goes negative in the middle of its support, thereby invalidating this solution.
But $g_2 \equiv 1.41088 \cdots < g_3 \equiv 1.64939\ldots$.
What happens in interval (g_2, g_3)?
Guess: Two support solution, supported on $[0, L_1)$ and $(L_2, 1]$, with $L_1 < L_2$
(roughly: negative part of $\rho^\star_{II}$ chopped off)

Guessing the form of $\rho^\star_{III}$ in this regime

$$\rho^\star_{III}(\xi) = \frac{B}{\sqrt{\xi(1-\xi)}}\sqrt{(\xi - L_1)(\xi - L_2)^3}\,\frac{\xi + D}{1+\xi} \tag{8}$$

where B, D, L_1, L_2 are constants to be determined.
Logic: $\rho^\star_{II}$ at $g_{NS} = g_2$ has this form with $L_1 = L_2$
$\rho^\star_{IV}$ at $g_{NS} = g_3$ has this form with $L_1 \to 0$
Simplest "interpolation" between these limits

Fixing the constants

Define

$$F(z) = \frac{1}{1+z} + 2C_1 z + \pi \frac{B}{\sqrt{z(z-1)}} \sqrt{(z-L_1)(z-L_2)^3} \frac{z+D}{1+z}$$

F has imaginary part only on real intervals $(0, L_1)$ and $(L_2, 1)$
For z on these real intervals, real part is exactly L.H.S of our integral equation for $\rho^\star$.
So F has a chance of being expressed as

$$F(z) = \int_{-\infty}^{\infty} dx' \frac{\rho^\star(x')}{z - x'}$$

For this to work, $F(z) \sim 1/z$ for large $|z|$.
Fix constants by setting coefficients of z^1, z^0 to zero, and coefficient of z^{-1} to 1!

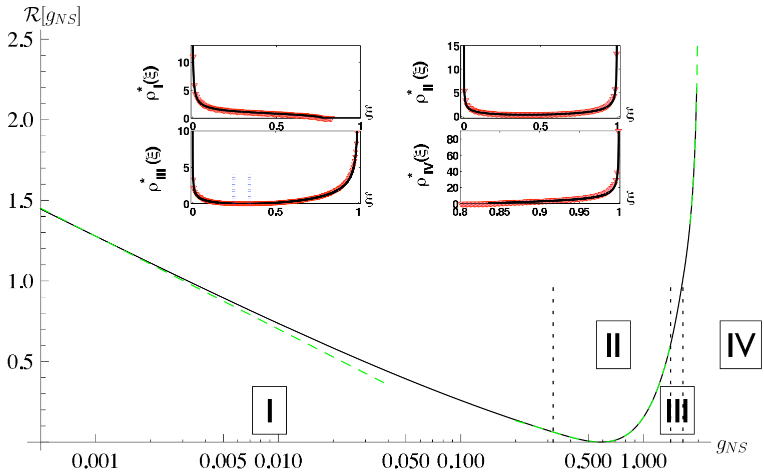
Summary: Form of solution

$$\mathcal{P}(G_{\rm NS}, N_c) \approx \exp\left[-N_c^2 \underbrace{\left(\mathcal{S}[\rho^\star] - \Omega_0\right)}_{\mathcal{R}(g_{\rm NS})}\right].$$

$$\rho^\star(\xi) = \begin{cases} \rho^\star_I(\xi) & \text{for } g_0 = 0 \le g_{\rm NS} \le g_1, \\ \rho^\star_{II}(\xi) & \text{for } g_1 \le g_{\rm NS} \le g_2, \\ \rho^\star_{III}(\xi) & \text{for } g_2 \le g_{\rm NS} \le g_3, \\ \rho^\star_{IV}(\xi) & \text{for } g_3 \le g_{\rm NS} \le g_4 = 2, \end{cases}$$

where $g_1 \equiv 2 - 19/8\sqrt{2} = 0.320621\ldots$,
$g_2 \equiv (968 - 499\sqrt{2} + 102\sqrt{17})/484 = 1.41088\ldots$ and
$g_3 \equiv 2 - (9 - \sqrt{21})/\sqrt{15(6 + \sqrt{21})} = 1.64939\ldots$.

More pictorially:



Result

- $P(G_{NS})$ has central Gaussian region with known variance
- Marked asymmetry in the large-deviation asymptotics near $G_{\rm NS} \to 0$ where $\mathcal{P}(G_{\rm NS}, N_c) \sim g_{\rm NS}^{N_c^2/4}$ and near $G_{\rm NS} \to 2N_c$ where $\mathcal{P}(G_{\rm NS}, N_c) \sim (2 - g_{\rm NS})^{N_c^2/2}$.
- Contrast with symmetric large-deviation tails for usual G

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