Thermodynamic signature of two-step melting of $\sqrt{3} \times \sqrt{3}$ order

Singular ferromagnetic suscepbility of the triangular lattice transverse-field Ising antiferromagnet

Kedar Damle, Quantum Magnetism Workshop, SJTU 2019

TIFR Mumbai

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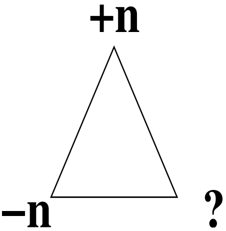
Geometric frustration of exchange interactions

Triangles on my mind...



- Triangles *frustrate* Néel order
- Geometry ☞ *competition* between leading exchange interactions

Frustration spawns novel states

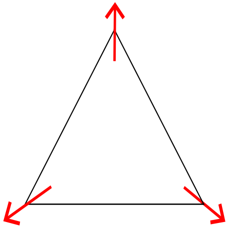
- Quenching of leading J ☞ *J cannot pick ground state at classical level*
- Sub-dominant interactions & quantum fluctuations ☞ *Variety of novel low temperature states*

Single ion anisotropy can be large

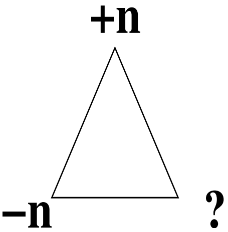
- Single ion anisotropy $-D(\mathbf{S}\cdot\mathbf{n})^2$can dominate over J
- Pyrochlore *spin ice* $Ho_2Ti_2O_7$ (Ho^{3+}, $(L+S)=8$)
  Easy axes **n** point outward from center of each tetrahedron
  $D \sim 50K$, $J \sim 1K$
  Harris *et. al.*, Phys. Rev. Lett. 79, 2554 (1997)
- Kagome Nd-langasite $Nd_3Ga_5SiO_{14}$ (Nd^{3+}, $(L+S)=9/2$)
  Easy axis perpendicular to lattice plane, $J \sim 2K$, $D \sim 10K$
  Robert *et. al.*, Physica B 2006
- $J \ll D$ is classical
  ☞ study leading quantum effects in a J/D expansion

Anisotropy amplies frustration

- Isotropic spins on a triangle



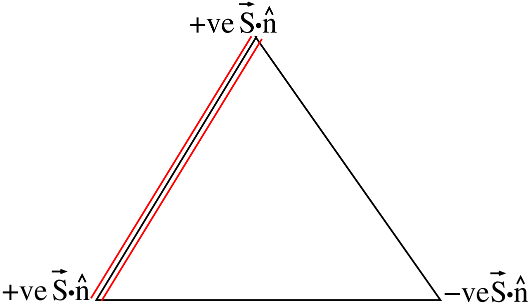
- Easy-axis **n** and triangular motifs...



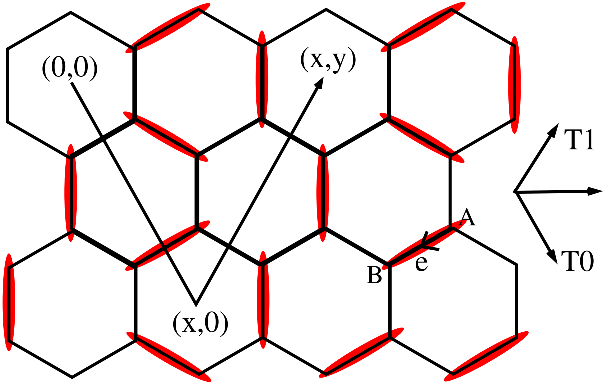
Wannier's triangular lattice model

- $H = J\sum_{\langle ij\rangle} \mathbf{S}_i \cdot \mathbf{S}_j - D\sum_i (S_i^z)^2$, with $D >> J$ on the triangular lattice.
- To leading order $S_i^z = \pm S \to \sigma = \pm 1$
  $H \approx JS^2 \sum_{\langle ij\rangle} \sigma_i \sigma_j$
- Minimum energy configurations?

Minimally frustrated configurations



- One frustrated bond per triangle
- Honeycomb lattice dimer model (one dimer touching each honeycomb site)



Honeycomb lattice dimer model: One dimer touching each honeycomb vertex

Classic problem in graph-theory/combinatorics/statistical mechanics

Ising 'liquid' in $T \to 0$ limit

- Calculation of Stephenson (64) gives

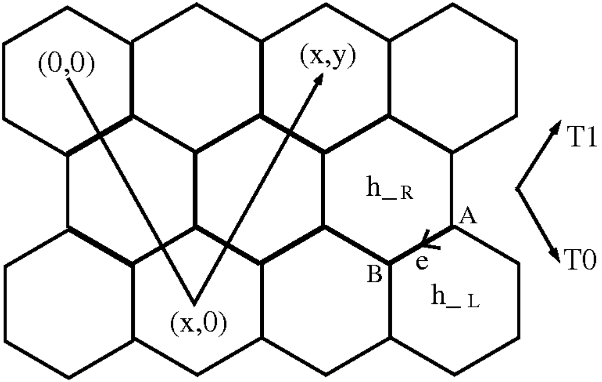
$$\langle \sigma(\mathbf{r})\sigma(0)\rangle \quad \sim \quad \frac{A}{r^{9/2}} + \frac{B\cos\left(2\pi(x+y)/3\right)}{\sqrt{r}}$$

- Spins neither freeze, nor fluctuate independently.
- Instead, form highly correlated "spin liquid".

Understanding this result:

- Dimers, heights, and Ising models of frustration
- (Obvious) connection to odd Ising gauge theories
- Connection to Kosterlitz-Thouless theory

Spins to dimers to electric fields



"Electric field" $\mathbf{e}_{\mathrm{A}\to\mathrm{B}} = n_{AB} - 1/3$

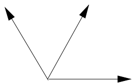
dimer constraint: Gauss law (!).

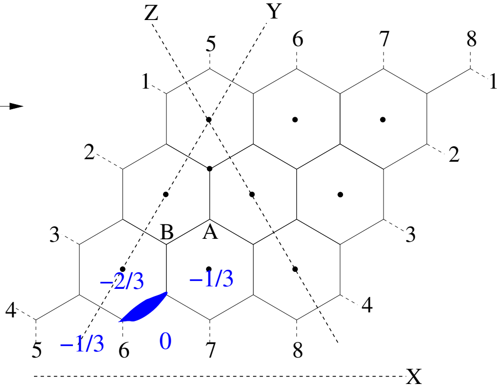
Youngblood and Axe (1980)

From dimers to microscopic heights $H(R)$

$$e_l = H_{L(l)} - H_{R(l)} \tag{1}$$

height field H on the original triangular lattice sites R





Henley 1990s

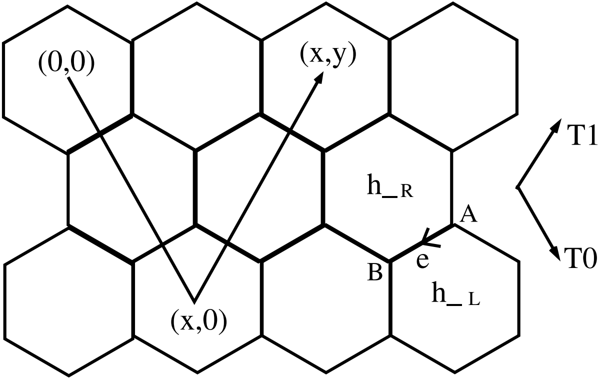
From microscopic $H(R)$ to coarse-grained $h(r)$

- Coarse-grain: Average over local rearrangements
- Locality: What happens "outside" cannot affect what happens "inside". $h(r) \to h(r) + 1$
  (Field theorists: "compactification radius")
- Lattice translations and $2\pi/6$ rotation ☞
  $h(r) \to h(r) + 1/3$, $h(r) \to -h(r)$

Ising spins in terms of $h(r)$

$3H$ jumps by odd (even) number whenever one crosses an unfrustrated (frustrated) bond

$\sigma(R) = \exp(-3\pi i H(R))$ (if $\sigma(R=0) = +1$, and $H(R=0) = 0$)



Dimer crossed $\rightarrow$ spin unchanged; empty link $\rightarrow$ spin flipped

$\sigma(R) = \exp\left(i\pi \sum_l (1 - n_l)\right) \sigma(0)$

$\sigma(R) = \exp(\frac{2\pi}{3} i(X + Y) - i\pi H(R))$

Fradkin et al (2004), KD (2009)

Effective action and operators

- Fewer flippable plaquettes $\rightarrow$ larger "tilt"
  $\mathcal{S}_{\text{eff}} = \frac{\pi}{g}(\nabla h)^2 + \lambda_6 \cos(6\pi h) + \dots$
  Coarse-grained representation of spins:
  $\sigma(r) \sim A e^{i\mathbf{Q}\cdot r} e^{-i\pi h(r)} + B e^{-3i\pi h(r)} + h.c.$
  Three-sublattice order parameter $\psi \sim e^{i\pi h} \equiv e^{i\theta}$ (!)

$T > 0$: Odd Ising gauge theory and Kosterlitz-Thouless vortices

- Nonzero temperature: Fully frustrated triangle $\rightarrow$ three dimers touching honeycomb site.
- "Electric field $E_{\mathrm{A}\rightarrow\mathrm{B}} = n_{AB} - 1/3$ no longer divergence-free
  But violations are 0 mod 2
  Field-theory language: Configuration space of odd-Ising gauge theory
- Heights no longer single valued
  Three dimers touching honeycomb site $\rightarrow$ vortex/antivortex in $\theta = \pi h$
- $T = 0$: Vortex-free *xy* model for θ with 6-fold anisotropy
  $T > 0$: Vortices allowed

Picture for $T = 0$ power-law ordered phase

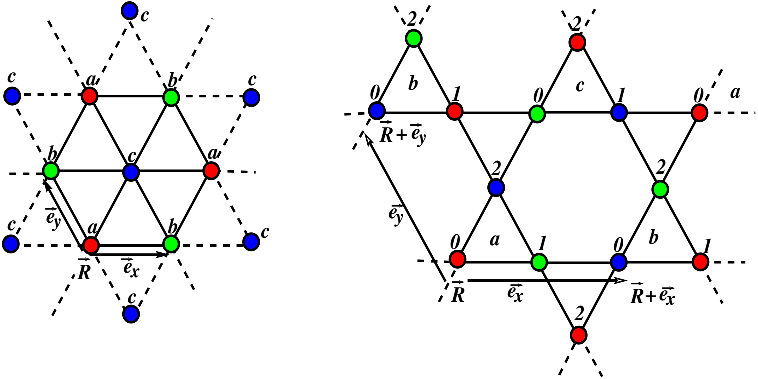
- In state with long-range three-sublattice order, θ feels $\lambda_6 \cos(6\theta)$ potential.
  Locks into values $2\pi m/6$ (resp. $(2m+1)\pi/6$) in ferri (resp. antiferro) three-sublattice ordered state
- In power-law three-sublattice ordered state λ_6 does not pin phase θ
  θ spread uniformly $(0, 2\pi)$
- But vortices absent.

RG description

- Fixed point action: $S = \frac{1}{4\pi g}(\nabla\theta)^2$
- For $g > \frac{1}{9}$
  $\lambda_6 \cos(6\theta)$ *irrelevant* along fixed line
  ☞ $\langle\psi^*(r)\psi(0)\rangle \sim \frac{1}{r^{\eta(T)}}$ with $\eta = g$
  Relies on absence of vortices at $T = 0$

Jose, Kadanoff, Kirkpatrick, Nelson (1977)

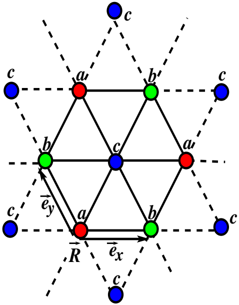
Easy-axis antiferromagnets on triangular lattices



Natural tripartite structure ☞

Perturbations/quantum fluctuations easily stabilize this order...

Three-sublattice order on the triangular lattice



θ	●	●	●
0	–S	+S	+S
π/6	–S	0	+S

$\psi = |\psi| e^{i\theta} = -\sum_{\vec{R}} e^{i\mathbf{Q}\cdot\vec{R}} S^z_{\vec{R}}$

Ferri vs antiferro order distinguished by the choice of phase θ

Ferri: $\theta = 2\pi m/6$, Antiferro: $\theta = (2m+1)\pi/6$ ($m = 0, 1, 2 \ldots 5$)

Very recent: $TmMgGaO_4$ (Thulium [Xe] $4f^{13}$ $6s^2$) Tm^{3+} non-Kramers

$J_z = \pm 6$

Princeton, Augsburg, Fudan (2017-18)

cf: talk by Liu (Gang Chen group) in this workshop

Intrinsic transverse field $\Gamma_{\rm int}$*:* $\sigma^x_{\rm eff}$ *not a dipole*

Prototypical example of order-by-(quantum) disorder

- $H_{\mathrm{TFIM}} = J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - \Gamma \sum_i \sigma_i^x$ on the triangular lattice
  small Γ ☞ Long-range order at three-sublattice wavevector **Q**
- Ordering of "antiferro" type $\rightarrow (+, -, 0)$
  antiferro order provides maximum "room" for quantum fluctuations
  Moessner, Sondhi, Chandra (2001), Isakov & Moessner (2003)

Another example: $S = 1$ with easy-axis single-ion anisotropy

- $H_{\rm AF} = J\sum_{\langle ij\rangle} \vec{S}_i \cdot \vec{S}_j - D\sum_i (S^z_i)^2$ on triangular lattice
  Closely related to effective model for AgNiO2
  (Seabra & Shannon '11)
- Low-energy physics for $D \gg J$:
  $H_b = -\frac{J^2}{D}\sum_{\langle ij\rangle}(b^\dagger_i b_j + h.c.) + J\sum_{\langle ij\rangle}(n_i - \frac{1}{2})(n_j - \frac{1}{2}) + \dots$
  KD & Senthil '06
- Low-temperature state for $D \gg J$: "supersolid" state of hard-core bosons at half-filling.
  Auerbach & Murthy (97), Heidarian & KD, Melko, Wessel...(05)
- Implies: ferri three-sublattice order in S^z + "ferro-nematic" order in $\vec{S}^2_\perp$
  (Simple easy-axis version of Chandra-Coleman (1991) "spin-nematic" ideas)

Symmetry breaking transitions: Generalities

- Symmetry-breaking state characterized by long-range correlations of "order-parameter" $\hat{O}$
- phenomenological Landau free energy density $\mathcal{F}[\hat{O}]$
  Expanding $\mathcal{F}$ in powers of $\hat{O}$ (symmetry allowed terms)
- Neglecting spatial variation & fluctuations:
  phase transition $\rightarrow$ change in minimum of $\mathcal{F}$

Fluctuation effects at continuous transitions:

- More complete description of long-wavelength physics:
  Include (symmetry allowed) gradient terms in $\mathcal{F}$
  Integrate over all possible order parameter configurations
- In most cases: Corrections to mean-field exponents

Symmetries are (usually) decisive:

- Transformation properties of $\hat{O}$ determine nature of continuous transition

Landau-theory for melting of three-sublattice order

- $\mathcal{F} = K|\nabla\psi|^2 + r|\psi|^2 + u|\psi|^4 + \lambda_6(\psi^6 + \psi*^6) + \dots$
  Connection with six-state clock models
  $Z = \sum_{\{p_i\}} \exp[\sum_{\langle ij \rangle} V(\frac{2\pi}{6}(p_i - p_j))]$
  Each $p_i = 0, 1, 2, ...5$
  $V(x) = K_1 \cos(x) + K_2 \cos(2x) + K_3 \cos(3x)$
  Cardy (1980)

Simplest lattice model

$$H_{\rm xy} = -J_{\rm xy} \sum_{\langle \vec{r}\vec{r}' \rangle} \cos(\theta_{\vec{r}} - \theta_{\vec{r}'}) - h_6 \sum_{\vec{r}} \cos(6\theta_{\vec{r}}) \ .$$

(higher harmonics $J^{(p)}$ ($p = 2, 3$) left out of $H_{\rm xy}$ for simplicity)

Melting scenarios for three-sublattice order

- Analysis (Cardy 1980) of generalized six-state clock models
  $\rightarrow$ Three generic possibilities of relevance here:
  Two-step melting, with power-law ordered intermediate phase
  OR
  3-state Potts transition to ferromagnetic phase followed by loss of ferromagnetism via Ising transition at higher temperature.. or vice-versa...
  OR
  First-order transition (always possible!)

Melting of three-sublattice order in various examples

- Antiferro three-sublattice order in triangular lattice transverse field Ising model
  Two-step melting
  (Isakov & Moessner '01)
- Ferrimagn. three-sublattice order in triangular lattice-gas models of monolayer films
  Two-step melting
  D.P. Landau '83
- Ferri. three-sublattice order in Kagome Ising antiferromagnets
  With second-neighbour ferro couplings: Two step melting
  Wolf & Schotte '88
  With long-range dipolar couplings: Three-state Potts transition
  Moller & Moessner '09, Chern, Mellado, Tchernyshyov '11

Detecting power-law order?

Need scattering experiment to detect power-law version of Bragg peaks

Or

Real-space data by scanning some local probe $+$ Lots of image-processing

Alternate thermodynamic signature(!)

- Singular thermodynamic susceptibility to *uniform* easy-axis field B:
  $\chi_u(B) \sim \frac{1}{|B|^{p(T)}}$
- $p(T) = \frac{4-18\eta(T)}{4-9\eta(T)}$ for $\eta(T) \in (\frac{1}{9}, \frac{2}{9})$
  So $p(T)$ varies from $2/3$ to 0 as T increases from T_{c1} to just below T_{c2}

(KD PRL '15)

Recall: picture for power-law ordered phase

- In state with long-range three-sublattice order, θ feels $\lambda_6 \cos(6\theta)$ potential.
  Locks into values $2\pi m/6$ (resp. $(2m+1)\pi/6$) in ferri (resp. antiferro) three-sublattice ordered state for $T < T_{c1}$
- In power-law three-sublattice ordered state for $T \in (T_{c1}, T_{c2})$, λ_6 does not pin phase θ
  θ spread uniformly $(0, 2\pi)$
- But vortices continue to be irrelevant
  Distinction between ferri and antiferro three-sublattice order lost for $T \in (T_{c1}, T_{c2})$
  Ferromagnetic response part of the time...

Recall: More formally

- Fixed point free-energy density: $\frac{\mathcal{F}_{\mathrm{KT}}}{k_B T} = \frac{1}{4\pi g}(\nabla\theta)^2$
  with $g(T) \in (\frac{1}{9}, \frac{1}{4})$ corresponding to $T \in (T_1, T_2)$
- $\lambda_6 \cos(6\theta)$ *irrelevant* along fixed line
- $\langle \psi^*(r)\psi(0)\rangle \sim \frac{1}{r^{\eta(T)}}$
  with $\eta(T) = g(T)$

Jose, Kadanoff, Kirkpatrick, Nelson (1977)

General argument—I

Starting point: Ferrimagnetic three-sublattice order also involves uniform magnetization m

More complete theory should treat m and ψ on equal footing

- Symmetries allow coupling term $\tilde{\lambda}_3 m(\psi^3 + \psi*^3)$
  augment $\frac{\mathcal{F}_{\rm KT}}{k_B T}$ with gapped free-energy density $\mathcal{F}_{\rm ferro}(m)$:
  $\mathcal{F}_{\rm ferro}(m) + \lambda_3 m \cos(3\theta)$
- λ_3 formally irrelevant along fixed line $\mathcal{F}_{\rm KT}$
  $\rightarrow$
  Physics of two-step melting unaffected—m "goes for a ride..."

But ...

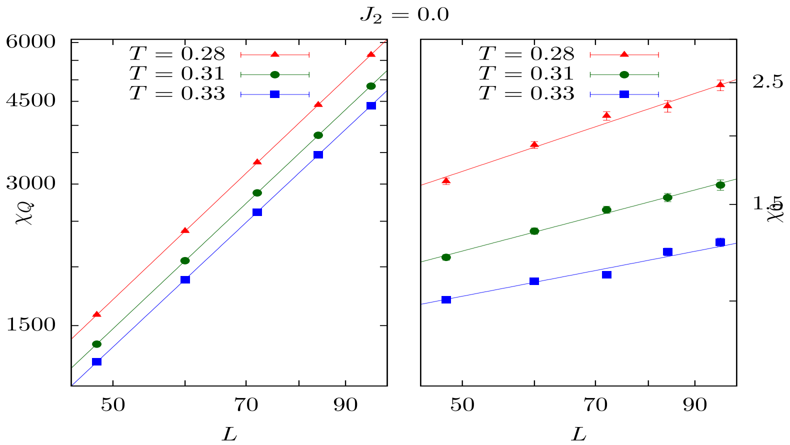
General argument—II

- m "inherits" power-law correlations of $\cos(3\theta)$:
  $C_m(r) = \langle m(r)m(0)\rangle \sim \frac{1}{r^{9\eta(T)}}$
- $\chi_L \sim \int^L d^2 r C_m(r)$ in a finite-size system at $B = 0$
- $\chi_L = \chi_{\mathrm{reg}} + bL^{2-9\eta(T)}$ for $\eta(T) \in (\frac{1}{9}, \frac{2}{9})$
  Diverges with system size at $B = 0$

General argument—III

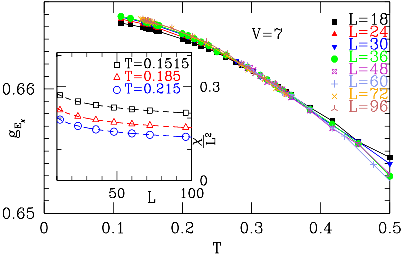
- Uniform field $B > 0 \rightarrow$ additional term $h_3 \cos(3\theta)$ in $\mathcal{F}_{\mathrm{KT}}$
- Strongly relevant along fixed line, with RG eigenvalue $2 - 9g/2$
- Implies finite correlation length $\xi(B) \sim |B|^{-\frac{2}{4-9\eta}}$
- $\chi_u(B) \sim |B|^{-\frac{4-18\eta}{4-9\eta}}$ for $\eta(T) \in (\frac{1}{9}, \frac{2}{9})$

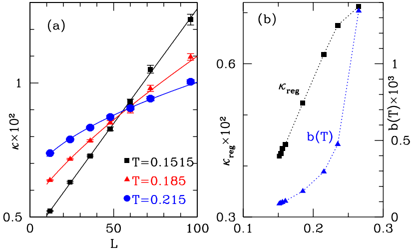
Test in prototypical example



In power-law ordered phase of H_{TFIM} on triangular lattice (Biswas & KD PRB '18)

Test in KT phase of easy-axis $S = 1$ triangular lattice AFM





In power-law ordered phase of H_b

(Heidarian & KD EPJB '18)

Also interesting:—Multicritical melting

- KT phase can
  Pinch-off at multicritical point $\mathcal{M}_?$, giving way to three-state Potts criticality. $c_{\mathcal{M}_?} =?$
  OR
  Pinch-off at multicritical point $\mathcal{M}_{\mathrm{Clock}}$, giving way to first-order transition line.
  (KD PRL '15)
- $\mathcal{M}_{\mathrm{Clock}}$ previously known, not $\mathcal{M}_?$
  Note: Conjecture (Dorey-Tateo-Thompson '96) relates $\mathcal{M}_{\mathrm{Clock}}$ to self-dual Z_6 $c = 1.25$ CFT (Zamolodchikov-Fateev '85)
  $\rightarrow c_{\mathcal{M}_{\mathrm{Clock}}} = 1.25$
  (KD PRL '15)

Also interesting: Multicritical melting

- How does the KT phase pinch-off for specific cases?
  - Evidence for $\mathcal{M}_{\mathrm{Clock}}$ on the triangular lattice
    (Rakala, Shivam, Desai, & KD *in prep.*)
  - Similar results on Kagome lattice systems
    Conjecture for $\mathcal{M}_{?}$ in triangular bilayers
    (Rakala & KD unpublished)

Acknowledgements and references

Students: D. Heidarian, G. Rakala, S. Biswas, S. Shivam, N. Desai

Resources: HPC facilities at DTP TIFR

References:

G. Rakala, S. Shivam, N. Desai, & KD *in prep.*

D. Heidarian & KD, Eur. Phys. J. B **91**, 202 (2018)

S. Biswas, KD PRB **97**, 085114 (2018)

KD, PRL **115**, 127204 (2015)

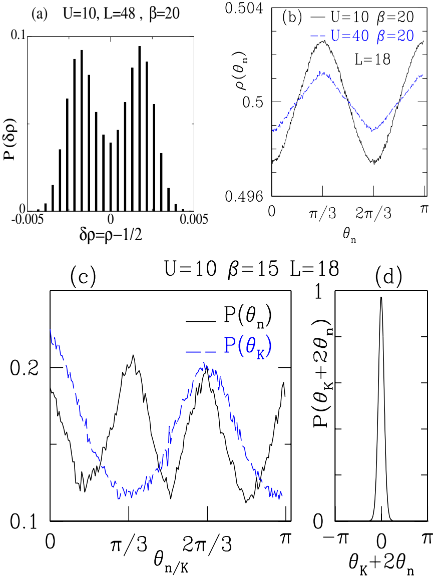
Some loose ends—I

Some loose ends...Three sublattice order and its melting in S=1 easy axis triangular antiferromagnet, and in classical Ising models on the triangular lattice

Is three-sublattice ordering of S^z in H_{AF} ferri or antiferro?

- Natural expectation: Quantum fluctuations induce antiferro order (like in the transverse field Ising model)
  $\rightarrow$
  Initial confusion: Ordering will be antiferro three-sublattice order
  e. g. Melko *et. al.* (2005)

Actual state has ferrimagnetic three-sublattice order

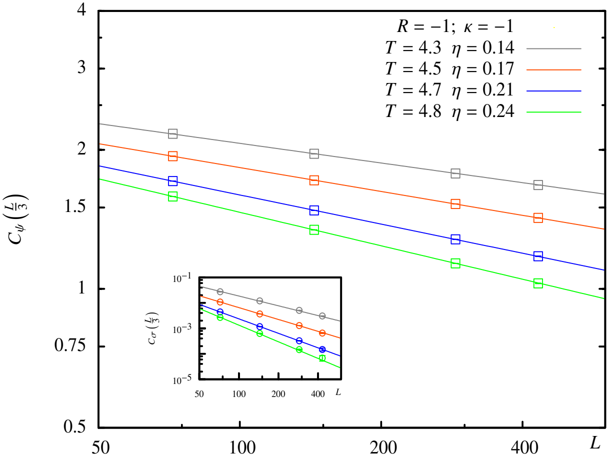


Heidarian & KD PRL ’05

Early work: Triangular lattice-gas models for monolayer films on graphite

- Three-sublattice long-range order of noble-gas monolayers on graphite
  Birgeneau, Bretz, Chan, Vilches, Wiechert...(1970—1990)
  $H_{\mathrm{J_1 J_2}} = J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - J_1 \sum_{\langle\langle ij \rangle\rangle} \sigma_i^z \sigma_j^z - J_2 \cdots - B \sum_i \sigma_i^z$
  Long-range three-sublattice ordering (wavevector **Q**) at low temperature
  D. P. Landau (1983)

Test in J1-J2model



In power-law ordered phase of $H_{\mathrm{J_1 J_2}}$

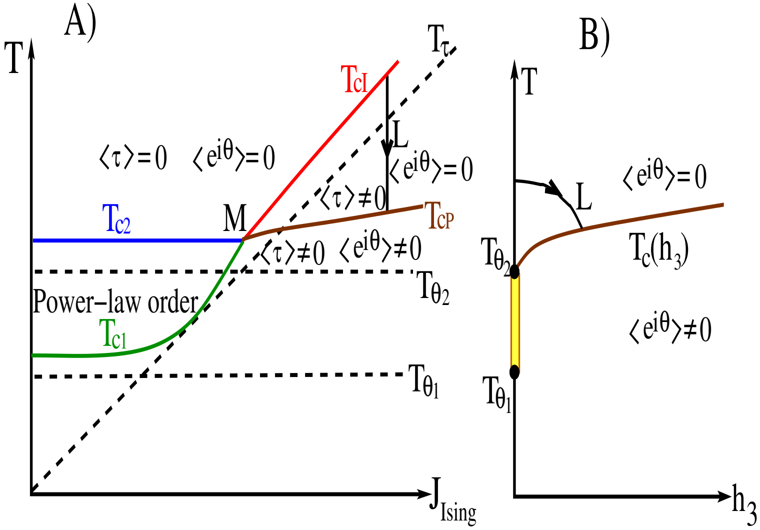
($R = -(J_1 + J_2)/J$ and $\kappa = (J_2 - J_1)/J$)

(Geet Rakala & KD *in prep.*)

Some loose ends—II

Some loose ends...multicritical melting

Multicritical melting of three-sublattice order



(KD PRL ’15)

More complete coarse-grained description

$$
\begin{aligned}
H_{\text{eff}} &= H_{\text{xy}} + H_{\text{Ising}} - J_{\theta\tau} \sum_{\vec{r}} \tau_{\vec{r}} \cos(3\theta_{\vec{r}}) \, , \\
\text{where} \quad H_{\text{Ising}} &= -J_{\text{Ising}} \sum_{\langle \vec{r}\vec{r}' \rangle} \tau_{\vec{r}} \tau_{\vec{r}'} - h \sum_{\vec{r}} \tau_{\vec{r}} \, , \\
H_{\text{xy}} &= -J_{\text{xy}} \sum_{\langle \vec{r}\vec{r}' \rangle} \cos(\theta_{\vec{r}} - \theta_{\vec{r}'}) - h_6 \sum_{\vec{r}} \cos(6\theta_{\vec{r}}) \, ,
\end{aligned}
$$

with $h \propto B$.
(KD PRL '15)

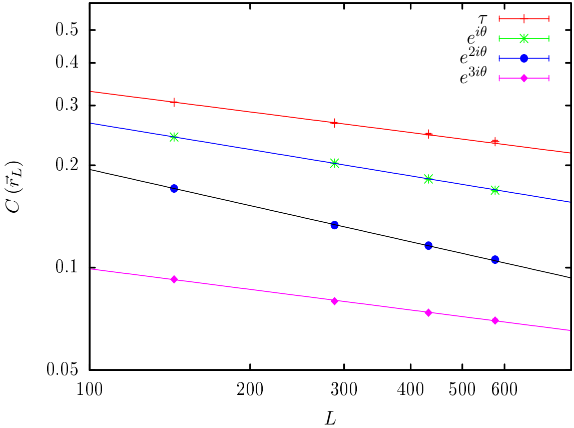
The argument...

- Start with known phase diagrams of H_{xy} and H_{Ising} and build in effects of $J_{\theta\tau}$
- When τ orders, H_{xy} sees effective three-fold symmetric perturbation $h_{3\text{eff}} \cos(3\theta_{\vec{r}})$ with $h_{3\text{eff}} \sim \langle \tau \rangle$
- When $e^{i\theta}$ orders, H_{Ising} sees effective field $h_{\text{eff}} \tau_{\vec{r}}$ with $h_{\text{eff}} \sim \langle \cos(3\theta) \rangle$

The "new" multicritical point $\mathcal{M}_?$

- c-theorem argument: $1 \leq c \leq \frac{3}{2}$
- To search:
  $J_{\mathrm{xy}} = h_6 = 1.0$, $J_{\theta\tau} = 0.25$
  Parametrize: $J_{\mathrm{Ising}} = f_{xy} T_{\theta_1} / T_\tau$ and $T = f_I f_{xy} T_{\theta 1}$ [with $T_{\theta 1} = 1.04$ and $T_\tau = 3.6409$]

Multicritical melting at $\mathcal{M}_?$



$[f_{\mathrm{xy}}^{\mathcal{M}_?}, f_I^{\mathcal{M}_?}] \approx [1.5570(8), 1.0061(5)]$

$C_{2\theta}$ [$C_{3\theta}$] rescaled by a factor of 7 [factor of 10]

$\eta_{3\theta} = \eta_\tau = 0.201(20)$, $\eta_\theta = 0.258(5)$, and $\eta_{2\theta} = 0.353(6)$.

(KD '15)

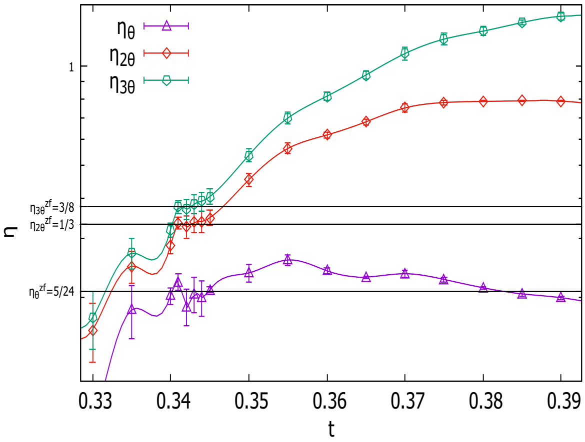
Speculation (aka wishful thinking?)

- If relative strength of first/second neighbour exchange tunable relative to long-range dipolar part in artificial kagome-ice: Could tune melting to multicritical point $\mathcal{M}_?$...
- Computations challenging due to long-range interactions

$\mathcal{M}_?$ vs $\mathcal{M}_{\rm clock}$

- Conjecture (Dorey '96): $\mathcal{M}_{\rm clock}$ corresponds to $c = 1.25$ self-dual Z_6 CFT constructed by Zamolodchikov-Fateev ('85).
- Conjecture yields exponents at $\mathcal{M}_{\rm clock}$: $\eta_{3\theta} = 3/8$, $\eta_{2\theta} = 1/3$, and $\eta_\theta = 5/24$.
  $\eta_{2\theta}$ and $\eta_{3\theta}$ very different from values at $\mathcal{M}_?$
  Recall: at $\mathcal{M}_?$, $\eta_{3\theta} = \eta_\tau = 0.201(20)$, $\eta_\theta = 0.258(5)$, and $\eta_{2\theta} = 0.353(6)$.

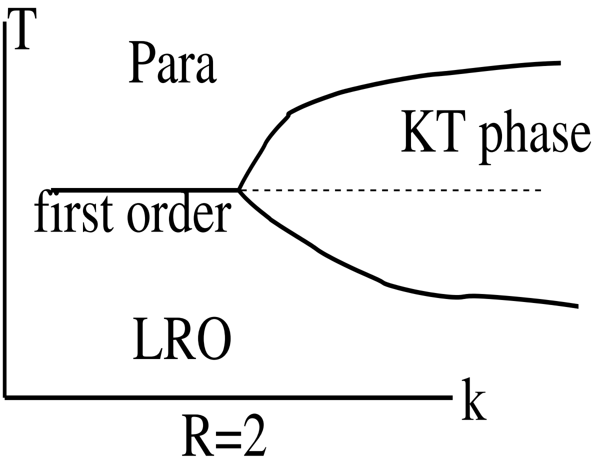
Test of conjectured exponents for $\mathcal{M}_{\text{clock}}$



Results on Cardy's six-state clock model

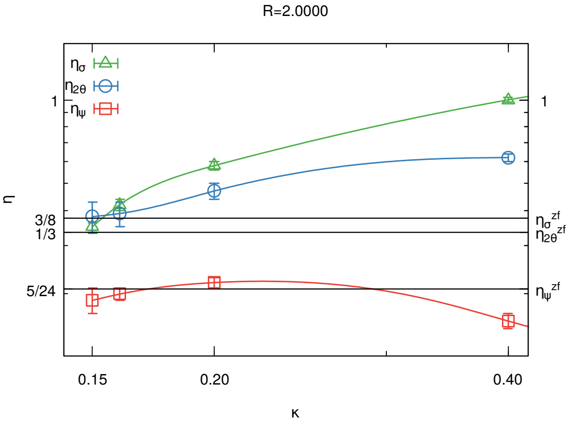
(Rakala, Shivam, & KD in prep.)

Schematic of pinch-off in triangular lattice Ising AFM



$J_1 = 1$, $R = J_2 + J_3$, $\kappa = J_2 - J_3$

Evidence for $\mathcal{M}_{\rm clock}$ in triangular Ising AFM



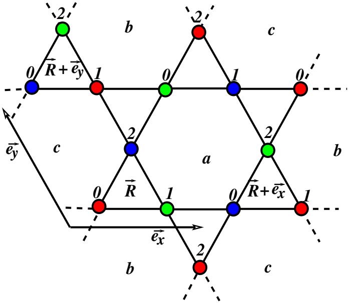
(Rakala, Shivam, & KD in prep.)

Some loose ends—III

Some loose ends...previous work on Kagome systems

Three-sublattice order on the Kagome lattice

θ	●	●	●
0	–S	+S	+S
$\frac{\pi}{6}$	–S	0	+S



$$\psi = |\psi| e^{i\theta} = -\sum_{\vec{R}} \sum_{\alpha=0,1,2} e^{i\mathbf{Q}\cdot\vec{R} - 2\pi i \frac{\alpha}{3}} S^z_{\vec{R},\alpha}$$

Again: Ferri vs antiferro distinguished by the choice of phase θ

Ferri: $\theta = 2\pi m/6$, Antiferro: $\theta = (2m+1)\pi/6$ ($m = 0, 1, 2 \ldots 5$)

Ising models for "Artificial Kagome-ice"

- $H_{\text{Kagome}} = J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - J_1 \sum_{\langle\langle ij \rangle\rangle} \sigma_i^z \sigma_j^z - J_2 \ldots$
- Only nearest-neighbour couplings $\rightarrow$ classical short-range spin liquid (Kano & Naya 1950)
- Second-neighbour ferromag. couplings destabilize spin liquid (Wolf & Schotte 88)
  Ferrimagnetic three-sublattice order at low T.
- "Artificial Kagome-ice: Moments $\mathbf{M}_i = \sigma_i^z \mathbf{n}_i$
  ($\mathbf{n}_i$ at different sites non-collinear)
  Expt: Tanaka *et. al.* (2006), Qi *et. al.* (2008), Ladak *et. al.* (2010,11)
  Theory: Moller, Moessner (2009), Chern *et. al.* (2011)