SERC lectures on large N Field theories

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Abstract

In these lectures we descibe various aspects of large N field theories.

Contents

1 Plan

`Lectures 1+2+3:`

A. The efficacy of large N: overview.

Central limit theorem, collective excitations tend to be classical.

1/N works as a small parameter in asymptotically free Yang-Mills gauge theory which otherwise do not have a free parameter (coupling merely sets the scale).

Large N factorization suggests that large N gauge theories may be described by a classical theory. Abstract writing of such a theory (effective potential of single traces)--- non-local. Is there a simple way to write such a theory? AdS/CFT is one.

Can large N can explain appearance of an arrow of time? Caldeira Legett model etc.

B. O(N) bosons:

Introduction of the (phi^2)^2 model in d dimensions. Lagrangian.

Large N classification of Feynman diagrams; reduction in the number of diagrams if we are interested in leading large N. Example: compute beta-functions in an epsilon expansion, Wilson Fisher fixed point.

Still infinite number of diagrams: how does one sum these?

Dyson Schwinger diagrammatic summation of bubble diagrams; gap equation.

Functional methods: Effective action in terms of the singlet variable. Large N as a saddle point expansion. 'Meson' fluctuations are suppressed. \sigma = \sigma_{cl} +

1/\sqrt N \sigma_q. Nearly free quasiparticles
<(\sigma_q)^3> \sim O(1/\sqrt N). 2->2 scattering \sim O(1/N)

Examine the classical effective potential of the meson. Can have symmetry breaking; Goldstones described by non-linear sigma model. Potential can also have a symmetric phase. Demarcation of phases in the (m^2, g) plane. Phase boundary= critical surface.

Critical phenomena. Large N gives exact critical exponents of Wilson-Fisher fixed point without epsilon expansion (\phi^4 in 3D). Comparison with experiments.

4+5+6.

C. O(N) fermions

Introduction of the Gross-Neveu model. Diagrammatics; bubbles (quick review of same stuff in Section B). Classical Symmetries: O(N), Chiral symmetry.

Impossibility of breaking chiral symmetry in perturbation theory.

Large N effective action. Exact beta function at large N: asymptotic freedom, dynamical mass generation, nonperturbative chiral symmetry breaking

Nambu Jona Lasinio model. Effective action of mesons. Comparison with BCS.

7+8+9.

D. Yang-Mills Gauge theories.

Repeat from Section A:
no tunable parameter in asymptotically free gauge theories,

as the coupling constant simply sets the scale.
Try 1/N as a possible perturbation parameter ('t Hooft).
[things are different in conformal theories, e.g. N=4 SYM,
where the YM coupling does not run and IS a tunable parameter;
however, even then, usual perturbation theory is
not powerful enough to arrive at many of the qualitative predictions
which large N perturbative expansion can.]

Double line notation. Planar vs non-planar Feynman diagrams.
N^(2-2*genus) expansion. At large N, only planar diagrams
survive (unless double scaling limit is taken).

Qualitative description of mesons and glueballs. Diagrammatics:
effective action of qqbar <JJ>, <JJJ> etc. Witten's argument that
J should be effectively described in terms of mesons. Observations:
(a) Infinite number of mesons
(b) <MMM> \sim O(1/\sqrt N), <MMMM> \sim O(1/N). Mesons are
good quasiparticles
(c) Zweig's rule: qqbar does not mix with qqqbarqbar, etc.

Introduction to large N baryons. Witten. Seiberg.

E. AdS/CFT

Large N factorization suggests that Large N YM should
be a classical theory, written in terms of the single trace variables.
The theory may involve Tr log, hence typically non-local.
(some of these are repeats from Section A.)

In AdS/CFT, the effective action in terms of
singlets of d-dim gauge theory becomes Einstein action
in D=d+1 dimensions! No proof exists (attempts at
proof below), but there are numerous evidences.

Parameter maps: [Recall G_N = l_s^8 g_s^2, \lambda = g_s N, R= l_s \lambda^
implying (in units R=1, or see p.59 MAGOO)

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G_N \sim 1/N^2
\alpha' \sim  1/\sqrt{\lambda}

Hence,
(N =\infty, \lambda=\infty) gauge theory = classical gravity in AdS
1/N expansion in gauge theory=  quantum gravity [\lambda=\infty]
1/\lambda expansion in gauge theory = classical string in AdS  [N=\infty]
finite N, finite \lambda= quantum string in AdS

Examples of calculations:
Free energy F(T, \lambda,  N) of {\cal N}=4 SYM in R^3 \times S^1.

Attempts at proof of AdS/CFT
O(N) bosons---> Vasiliev!

F. Matrix models at large N

reformulation in terms of eigenvalue
density (collective variables); proof of the existence of a large N
expansion.  Understanding in terms of Feynman diagrams.

phase transitions. Gross-Witten-Wadia critical point.

Double scaling. Genus sum can be regained if  (g-g_c)^a N is held
fixed.
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2 Efficacy of large N: An overview

2.1 Central Limit theorem (CLT)

Consider N *independent* random variables x_i

$$x_i = \mu_i + O(\sigma)$$

Define $X = 1/N \sum x_i$, $\mu = 1/N \sum \mu_i$.
CLT: $X = \mu + O(\sigma/\sqrt{N})$

Proof: [1] $\langle (X-\mu)(X-\mu)\rangle = 1/N^2 \sum_{ij}\langle (x_i-\mu_i)(x_j-\mu_j)\rangle = 1/N^2 \sum_i \langle (x_i-\mu_i)^2\rangle = \sigma^2/N$.

Lesson: fluctuation of average goes to zero in the limit $N \to \infty$. **collective excitations tend to be classical.**

Limitations of the proof: Variables are non-interacting.

To come: We will see, throughout these lectures, that if the averages are chosen *appropriately*, they become classical (= non-fluctuating), even if the random variables are interacting.

2.2 Infinite range Ising

(See Exercise 2 for more details)

Long-range Ising model

$$Z = \sum_{s_i} \exp[-H],\ H = -J\sum_{i,j} s_i s_j - h\sum_i s_i$$

$$Z = \int_{\infty}^{\infty} dm \exp[-m^2/2J + N\ln\cosh(m+h)]$$

In the 'tHooft limit $J = \tilde{J}/N$, we get a large N saddle point.

2.3 $O(N)$ vector models

Bosons:

Consider (3.1).

Define the magnetization $\sigma = \langle \phi_1(x)\rangle = \partial \ln Z[h_1(x)]/\partial h_1$.

We will see that, at large N (in the symmetry broken phase)

$$\sigma = \sigma_{cl} + O(1/\sqrt{N})$$

The generating function $\ln Z[h(x)]$ has a classical limit, with quantum flucuations suppressed by inverse powers of N. The functional integral is concen-

[1] In more detail: Redefine $x_i - \mu_i \to x_i, X-\mu \to X$. For simplicity, let x_i be a Gaussian variable with dispersion σ. $P(x_i) \propto \exp[-x_i^2/(2\sigma^2)]$. Joint probability $\prod_i P(x_i)dx_i$. Do an O(N) transformation of variables $x_1, x_2, .. \to y_1 = X\sqrt{N}, y_2, ...$ (note that $y_1 = \vec{n}_1.\vec{x}, \vec{n}_1 = 1/\sqrt{N}(1,1,...), |\vec{n}| = 1.$) The joint probability becomes $\prod_i P(y_i)dy_i$, with $P(y_1)dy_1 = \exp[-y_1^2/(2\sigma^2)]dy_1 = \exp[-X^2/(2\sigma^2/N)]dX\sqrt{N}$. Hence Proved.

trated around some classical value of σ. Large N factorization:

$$\langle\sigma(x)\sigma(y)...\rangle = \frac{\int \sigma(x)\sigma(y)... \exp[-NS(\rho)]}{\int \exp[-NS(\rho)]} = \sigma_{cl}(x)\sigma_{cl}(y)... + O(1/N) \tag{2.1}$$

$$\langle\sigma(x)\sigma(y)\rangle_c \sim 1/N$$

Fermions:

Consider (4.1).

In $d = 2$, this is the Gross-Neveu model. We will demonstrate that for the $O(N)$ singlet $\sigma = 1/N\ \bar{\psi}_i\psi_i$, for $g > 0$

$$\langle\sigma\rangle = \sigma_{cl} + O(1/\sqrt{N})$$

which exhibit asymptotic freedom, chiral sym. breaking and dynamical mass generation.

In $d = 4$, this is related to the Nambu Jona Lasinio model, which again shows chiral sym. breaking as above.[2]

2.4 Gauge theory

In YM theory, by dimensional transmutation

$$\Gamma^{(n)}(q_1\Lambda_{YM}, ..., q_N\Lambda_{YM}, g_1) = \Lambda_{YM}^{\#} f(q_1, ..., q_N) \tag{2.2}$$

Thus, the gauge coupling disappears, when $\Lambda_{YM} = 1$ units are used. There are no tunable parameters which can be tuned to a small value.

'tHooft suggested that in terms of appropriate (gauge-invariant) variables, $1/N$ for an $SU(N)$ gauge theory can be used as a small parameter.

Large N factorization suggests that large N gauge theories may be described by a classical theory. Abstract writing of such a theory (effective potential of single traces)— non-local. Is there a simple way to write such a theory? AdS/CFT is one.

Large N gauge theory has given rise to many beautiful things: a qualitative understanding of mesons and baryons, quantitative computability ['large D'], AdS/CFT,... [more]

[2]Such a theory is uv-incomplete and has to be regarded as an effective theory.

3 $O(N)$ bosons

We consider the following problem: calculate the functional integral

$$Z = \int D\phi \exp[-S] = \exp[-W]$$

$$S = \int d^d x[\frac{1}{2}\partial_\mu \phi_i^2 + \frac{1}{2}m^2\phi_i^2 + g/(4!N)(\phi_i^2)^2] \tag{3.1}$$

We will consider both $+$ and $-$ signs for m^2, however we will consider $g \geq 0$ since for negative g the potential will be bottomless and the theory will not have a stable vauum.

We will also be interested in

$$W[J] = \ln \int D\phi \exp[-S + \int J\phi_i^2] \tag{3.2}$$

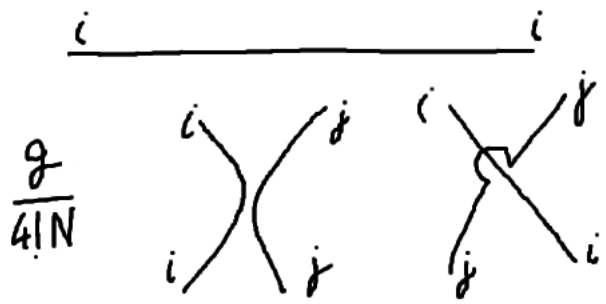
which generates

$$\langle \phi_i^2(x)\phi_i^2(y)...\rangle$$

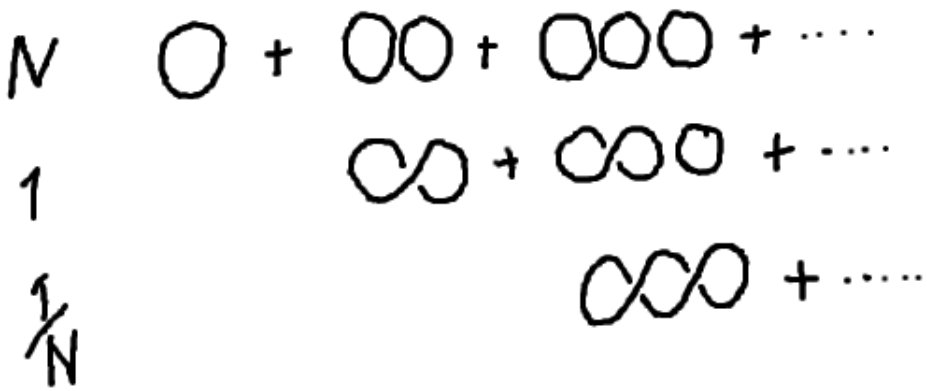
3.1 Feynman diagrams

Propagator and Vertex:

$$\langle \phi_i(p)\phi_j(-p)\rangle = \frac{1}{p^2+m^2}\delta_{ij}$$

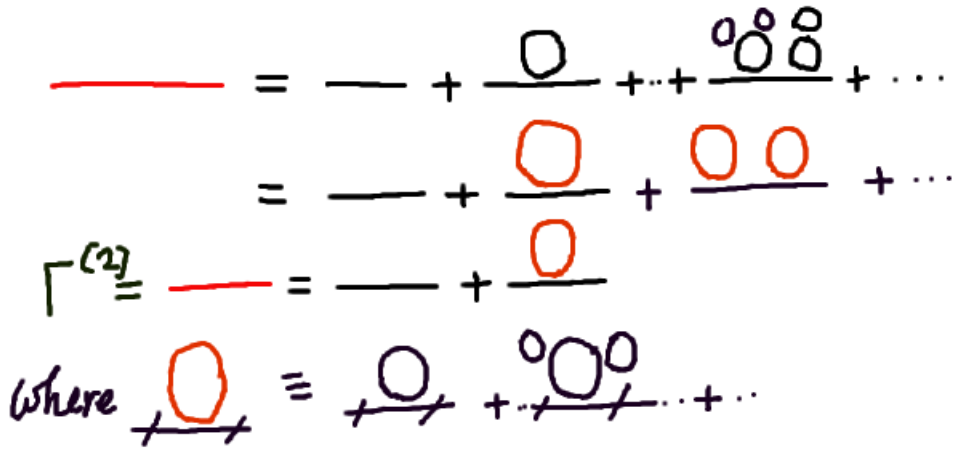


Consider the vacuum bubble diagrams contributing to W:



Clearly there is a lot of reduction in the Feynman diagrams if we are interested only in the leading large N contribution. However, there are still an infinite number of diagrams. But note the **self-repeating** pattern of the leading (namely $O(N)$) diagrams: it turns out to be summable. The subleading terms in the $1/N$ expansion do not have this **self-repeating** pattern.

Summing the leading diagrams by Dyson-Schwinger



Red propagator$= \frac{1}{p^2+m^2+\Sigma(p)}$ where the Self-energy $\Sigma(p)$ (actually independent of p) is given by

$$\Sigma = \Sigma = -1\text{PI contribution to}\Gamma^{(2)}(p) = g/6 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+m^2+\Sigma}$$

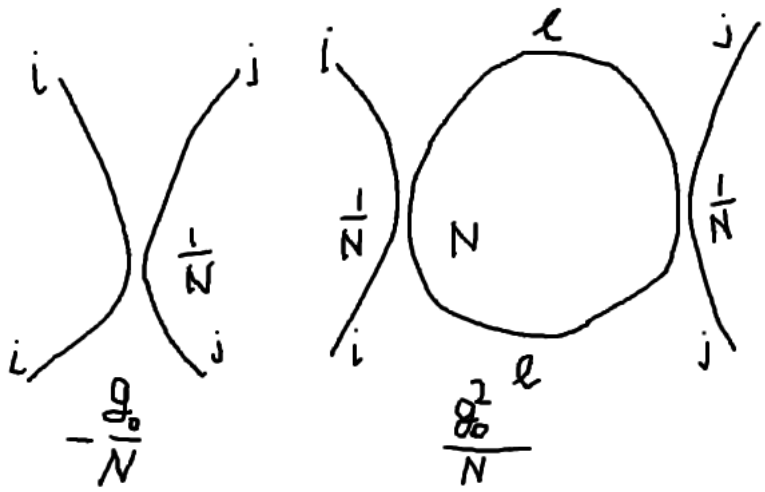
$$W = \text{Partition function with red propagator} = N \ln det(-\partial^2 + m^2 + \Sigma) + O(1) \tag{3.3}$$

$O(1)$ represents the non-leading diagrams ignored in this section.
Fun Exercise: Compute

$$1 + \frac{1}{1 + \frac{1}{1\frac{1}{1+..}}}$$

Can you see the similarity with the Dyson-Schwinger summation?

A sample calculation: beta function of g



$$\Gamma^{(4)}_{ijkr} = \frac{1}{3N}(\delta_{ij}\delta_{kl} + \text{two more terms})\left(-g_0 + \frac{g_0^2}{48\pi^2}\ln(\Lambda/m) + o(g_0)^3\right)$$

Must define $g_0(\Lambda)$ such that $\Gamma^{(4)}$ is independent of Λ (note no wavefunction renormalization to this order). Define $\beta_{g_0} = dg_0/d(\ln\Lambda)$.

Exercise Show that in $d = 4 - \epsilon$

$$\beta_{\bar{g}} = [-\epsilon\bar{g} + \frac{\bar{g}^2}{48\pi^2}(1 + O(1/N))](1 + O(\epsilon)) + O(g^3) \tag{3.4}$$

where $\bar{g} = g/\Lambda^{4-d}$ is the dimensionless coupling.

Remarks: (1) The correction factor is $(1 + 8/N)$.

(2) We obtain the beta-function by demanding that $\Gamma^{(4)}$ is Λ-dependent. Recall the Callan-Symanzik equation (in terms of bare quantities)

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - 4\gamma\right)\Gamma^{(4)} = 0, \text{ Hence} \tag{3.5}$$

In our case, since the self-energy $\Sigma(p)$ is p-independent (see the diagrams for $\Sigma(p)$ above), there is no wavefunction for ϕ, hence $\gamma = 0$. Hence the C-Z equation gives

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g}\right)\Gamma^{(4)} = 0$$

which is simply the statement that the (total) Λ-dependence of $\Gamma^{(4)}$ vanishes.

(3) **Wilson-Fisher fixed point**

Note that in (3.4) β vanishes in points $\bar{g} = 0$ and $\bar{g} = \bar{g}^*$, where

$$\bar{g}_* = \epsilon\ 48\pi^2(1 + O(1/N)) + o(\epsilon^2) \tag{3.6}$$

In general, the above is a double expansion in g, ϵ. Near the fixed point

Significance of a zero of β: At this and any other general example of a zero of beta-functions, the C-Z eq. (3.5) for the 2-pt function reduces to

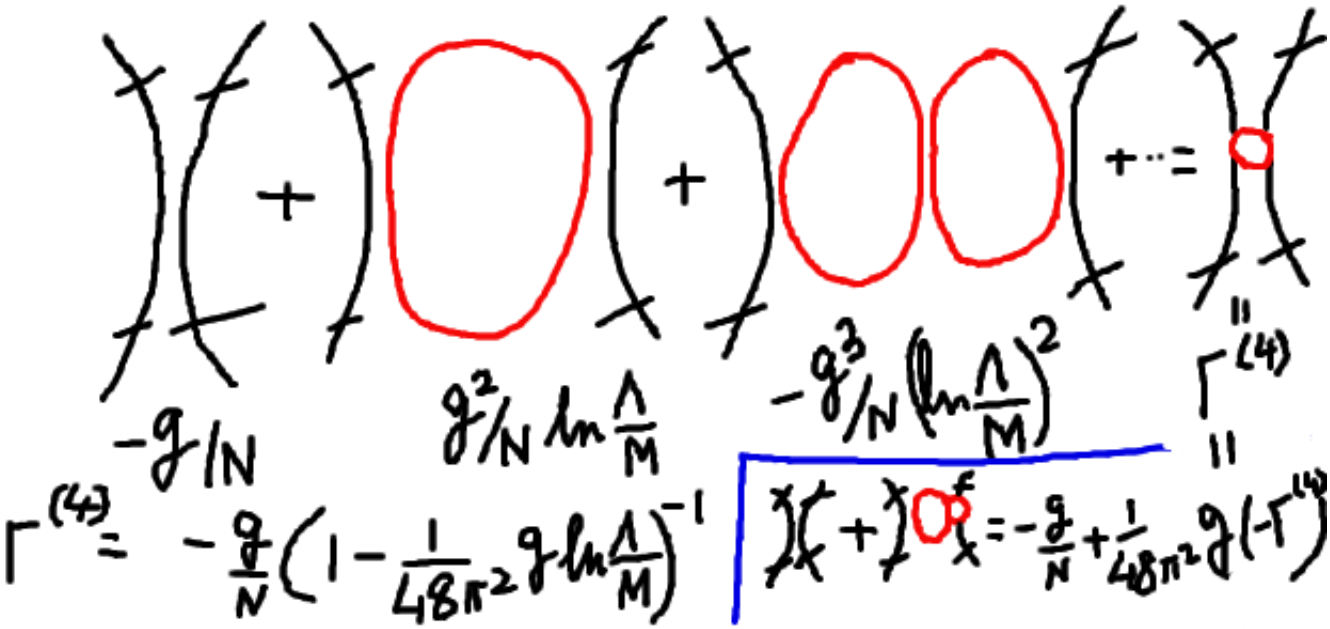
$$\left(\Lambda\frac{\partial}{\partial\Lambda} - 2\gamma(\bar{g}^*)\right)\Gamma^{(2)} = 0, \text{ Hence} \tag{3.7}$$

Using the representation $\Gamma^{(2)} = P^2 f(P/\Lambda)$ for momenta p_i of the scale P, the above equation implies a power law behaviour

$$\Gamma^{(2)} \sim P^{2-2\gamma(\bar{g}^*)}$$

which corresponds to a scale-invariant theory (typically, a CFT).

(4) Large N resummation:



Eqn. (3.4), for $\epsilon = 0$ (four dimensions) holds to all orders in g, in the large N limit.

3.2 Functional methods

Order parameter

Imagine applying a source (magnetic field) as follows:

$$Z[h(x)] = \int D\vec{\phi} \exp[-\int \frac{1}{2}(\partial\vec{\phi}^2 + m^2\vec{\phi}^2 + \frac{g}{24N}(\vec{\phi}^2)^2 + h(x)\phi_1(x)] \quad (3.8)$$

Clearly

$$\frac{\partial \ln Z}{\partial h(x)} = \langle \phi_1(x) \rangle$$

gives us the 'magnetization' (the O(N) symmetry breaking order parameter). Let us split the $\vec{\phi}$ as

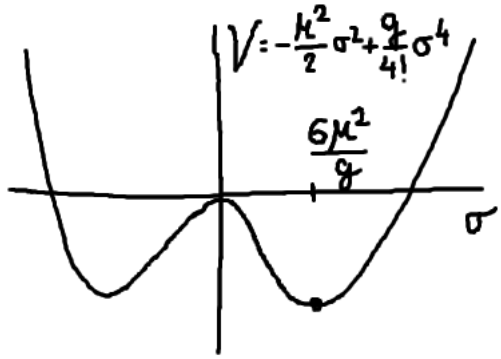
$$\vec{\phi} = (\phi_1 = \sigma, \phi'_i, i = 2, ...N)$$

Among other things we will be interested in the order parameter

$$M = \langle \sigma \rangle. \tag{3.9}$$

For $N = 1$, if $m^2 = -\mu^2 < 0$, we have a classical potential $V = -\mu^2/2\ \sigma^2 + g/24\ \sigma^4$, which classically allows for a non-zero magnetization:

$$\sigma^2_{class} = \frac{6\mu^2}{g} \tag{3.10}$$



However, the fluctuations are of the same order as the vev, and we can't be sure if they destroy the magnetic order or not.

We will see that this is where large N plays a role. It suppresses the fluctuation! (the σ is shifted from its classical value, however, see (3.19) and remarks below).

To begin seeing the effect of a large N, we would like to obtain the effect of all the ϕ''s on σ, by integrating out the ϕ'_i:

$$\int D\sigma D\phi'_i \exp[-S(\vec{\phi})] = \int D\sigma \exp[-\int [\frac{1}{2}(\partial\sigma^2 + m^2\sigma^2 + \frac{g}{4!N}\sigma^4] - (N-1)\Delta S_{eff}(\sigma)]$$

The spirit is like that a **Hartree method**. Because of the quartic interaction, we can't integrate out the ϕ' directly. We need to use auxiliary fields, which we will do next. Meanwhile a couple of exercises.

Exercise 1:

Replace

$$\phi_i\phi_i\phi_j\phi_j \to 2\langle\phi_i\phi_i\rangle[\phi_1^2+\phi_2^2+...] \equiv 2N\rho[\phi_1^2+\phi_2^2+...]$$

where we have assumed a spacetime-indepedent condensate

$$\langle\phi_i(x)\phi_i(x)\rangle = N\rho$$

Demand the self-consistency of the above eqn. and get the "gap" equation

$$\rho = \int_k \frac{1}{k^2+m^2+g/6\ \rho} \tag{3.11}$$

This is the same as the Dyson-Schwinger, with $\Sigma = \rho$. Also compare with (3.16).

Exercise 2:

Long-range Ising model

$$Z = \sum_{s_i} \exp[-H],\ \ H = -J\sum_{i,j} s_i s_j - h\sum_i s_i$$

Use the identity

$$\exp[J\sum_{i,j} s_i s_j] = \frac{1}{2\pi J}\int_{\infty}^{\infty} dm \exp[-m^2/2J + m\sum_i s_i]$$

We get

$$Z = \int_{\infty}^{\infty} dm \exp[-m^2/2J - \ln(2\pi J)] \prod_i \sum_{s_i} \exp[(m-h)s_i]$$

Compute the 'gap' equation for m. Compute Z.

Auxiliary fields

Introduce in the functional integral (3.1)

$$1 = \int D\rho\delta((N\rho - (\vec{\phi})^2)$$

so that

$$Z = \int D\rho D\vec{\phi} \exp[-\int d^d x \frac{1}{2}(\partial\vec{\phi})^2 + NU(\rho)]\delta(N\rho - (\vec{\phi})^2)$$

$$U(\rho) = m^2/2\rho + \frac{g}{4!}\rho^2$$

Use

$$\delta((N\rho - (\vec{\phi})^2) = \int D\lambda \exp[\int d^d x \lambda/2 \ (N\rho - (\vec{\phi})^2)]$$

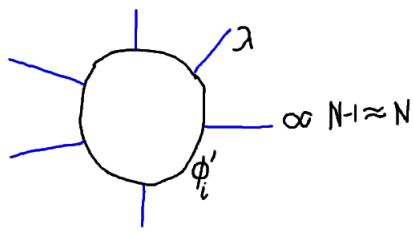
Contour of λ must be parallel to the imag axis. In the σ, ϕ' notation

$$Z = \int D\sigma D\phi'_i D\lambda D\rho e^{-S[\sigma,\phi'_i,\lambda,\rho]}$$

where

$$\begin{aligned} S[\sigma, \phi'_i, \lambda, \rho] &= \int d^d x \frac{1}{2}\phi'_i(-\partial^2 + \lambda)\phi'_i + \frac{1}{2}\sigma(-\partial^2 + \lambda)\sigma + N(U(\rho) - \lambda\rho/2) \\ &= (\mathbf{N-1})\frac{1}{2}Tr\ln(-\partial^2 + \lambda) + \int d^d x[\frac{1}{2}\sigma(-\partial^2 + \lambda)\sigma + N(U(\rho) - \lambda\rho/2)] \end{aligned} \tag{3.12}$$

In the second step, we have integrated out ϕ'. [Diagrammatically, this corresponds to a 1-loop diagram, with the ϕ'_i's running internally in the loop]. Note the appearance of the BIG colour factor **(N-1)**. For this, it is important to have the auxiliary field λ (dotted line) as an <u>$O(N)$ singlet</u>!



large N: Clearly, if there are saddle point solutions to σ, it will be

$O(\sqrt{N})$. Accordingly, let us scale $\sigma = \sqrt{N}\tilde{\sigma}$ [3] and (drop the tilde) to get

$$S = \mathbf{N}[\frac{1}{2}Tr\ln(-\partial^2+\lambda) + \int d^dx \frac{1}{2}\sigma(-\partial^2+\lambda)\sigma + U(\rho) - \lambda\rho/2] \tag{3.13}$$

Here we have used $N-1 \approx N$.

Spacetime independent saddle points

Look for spacetime independent saddle points.

$$V_{eff} = S/V_d = \mathbf{N}\left(\frac{1}{2}\lambda\sigma^2 + U(\rho) - \lambda\rho/2 + \frac{1}{2}\int_k [\ln(k^2+\lambda) - \ln(k^2)]\right)$$

In $d=3$,

$$V_{eff} = \mathbf{N}\left(\frac{1}{2}\lambda\sigma^2 + U(\rho) - \lambda\rho/2 + \frac{-6\lambda^{3/2}\tan^{-1}\left(\frac{\Lambda}{\sqrt{\lambda}}\right) + 3\Lambda^3\log\left(\frac{\lambda}{\Lambda^2}+1\right) + 6\lambda\Lambda}{36\pi^2}\right) \tag{3.14}$$

EOM (moduli space):

$$\begin{aligned} &\lambda\sigma = 0 \\ &U'(\rho) = m^2/2 + g/12\ \rho = \lambda/2 \\ &\sigma^2 - \rho + \int_k \frac{1}{k^2+\lambda} = 0 \end{aligned} \tag{3.15}$$

It is useful to note the gap equation:

$$\rho = \int_k \frac{1}{k^2+m^2+g/6\ \rho} + \sigma^2 \tag{3.16}$$

Note that the above equations also apply to $O(N)$ models with a general polynomial potential

$$U(\vec{\phi}) = \sum_n g_n/N^{n-1}\ (\vec{\phi}^2)^n, n = 2,3,4,...,\Rightarrow U(\rho) = \sum_n g_n\rho^n, n = 2,3,4,...$$

[3] It might appear strange that we are treating a single component of $\vec{\phi}$ as $O(\sqrt{N})$. However, this is familiar from SSB physics: in the SSB phase we can represent the classical vacuum as $\vec{\phi} = (\sigma, 0, 0, ...)$ where the particular orientation is chosen by some 'magnetic field' as in Eq. (3.8). In that case, since $\vec{\phi}^2/N \sim O(1)$, we must have $\sigma^2/N \sim O(1)$, hence $\sim O(\sqrt{N})$. See also (3.20).

Large N factorization

Note that (3.15) represent large N Saddle point hence the fluctuations around the saddle point solutions are down by $1/\sqrt{N}$.

$$\sigma = \sigma_{cl} + \frac{1}{\sqrt{N}}\delta\sigma$$

Hence,

$$S = N\bar{S}[\sigma_{cl}] + (\delta\sigma)^2 + \frac{1}{\sqrt{N}}(\delta\sigma)^3$$

In the presence of $h(x)$ (see (3.8), the classical saddle point value and the classical effective action, will both depend on h:.

$$\begin{aligned}
Z &= e^{NS_{saddle}[h(x)](1+O(1/N))} \\
\langle\sigma(x)\rangle &= e^{NS_{saddle}[h(x)]}\frac{\partial}{\partial h(x)}e^{-NS_{saddle}[h(x)]}(1+O(N)) = \frac{N\partial S_{saddle}[h]}{\partial h(x)}(1+O(N)) \\
\langle\sigma(x)\sigma(y)\rangle &= e^{NS_{saddle}[h(x)]}\frac{\partial}{\partial h(x)}\frac{\partial}{\partial h(y)}e^{-NS_{saddle}[h(x)]}(1+O(N)) \\
&= [\frac{N\partial S_{saddle}[h]}{\partial h(x)}\frac{N\partial S_{saddle}[h]}{\partial h(y)} + \frac{N\partial^2 S_{saddle}[h]}{\partial h(x)\partial h(y)}](1+O(1/N)) \\
&= \langle\sigma(x)\rangle\langle\sigma(y)\rangle(1+O(1/N)) \qquad (3.17)
\end{aligned}$$

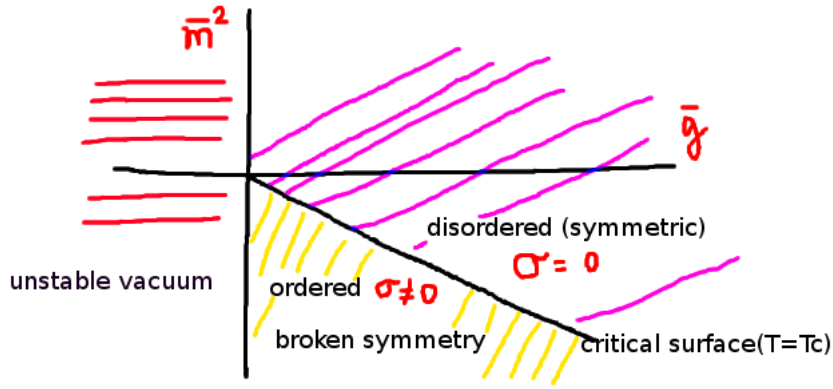
which proves the so-called large N factorization.

We will encounter a Diagrammatic proof later.

Phases

The first eqn. in (3.15) can be solved by

$$\begin{aligned}
&\lambda = 0, \sigma \neq 0 \quad \text{broken phase,} \quad \text{happens if } m^2 < -\mu_c^2,\ \mu_c^2 = g/6 \int_k \frac{1}{k^2} \\
&\lambda \neq 0, \sigma = 0, \quad \text{symmetric phase,} \quad m^2 > -\mu_c^2 \\
&\lambda = \sigma = 0 \quad \text{critical surface}, m^2 = -\mu_c^2 \qquad (3.18)
\end{aligned}$$



Broken phase

In the first case, possible if $m^2 = -\mu^2$=negative, the solution is:

$$\rho = -6m^2/g = 6\mu^2/g$$
$$\sigma^2 = \rho - \rho_c = 6/g\ (\mu^2 - \mu_c^2),\ 6/g\ \mu_c^2 \equiv \rho_c \equiv \int_k \frac{1}{k^2} \tag{3.19}$$

- Clearly we must have $m^2 < -\mu_c^2$. (recall we need $g > 0$ for vacuum stability)
- Note the shift in the quantum order parameter in (3.19) compared to (3.10). the shift, ρ_c measures the quantum fluctuations. Note that the extremal value of ρ coincides with (3.10) the extremum of the classical potential.
- Note the appearance of $N-1$ **Goldstones**, consistent with the breaking of $O(N) \to O(N-1)$. Roughly speaking, in (3.12) if $\lambda = 0$, the fields $\phi_i', i = 2, ..., N$ become massless. This argument is not rigorous, since naively it would seem that even the field σ in (3.12) would appear to be massless by the same token [4]. The correct argument, is to parametrize

[4] Actually, $\delta\sigma$ fluctuations couple to $\delta\lambda$ fluctuations, which leads to a non-trivial self-energy diagram and mass correction to $\delta\sigma$. The same thing does not happen to the π_i's described below.

the 'other' fields as

$$\vec{\phi} = \mathbf{g}(\pi_i).(\sigma, 0, 0, 0, ...) = \sigma \mathbf{g}(\pi_i).\mathbf{e_1} = (\sigma_0 + \delta\sigma)\, \mathbf{g}(\pi_i).\mathbf{e_1} \tag{3.20}$$

where

$$\mathbf{g}(\pi_i) = \exp[i \sum_{i=1}^{N-1} T_i \pi(x)] \ \in O(N)/O(N-1), \quad \mathbf{g} \simeq \mathbf{gh},$$

Thus, $\vec{\phi}^2 = (\sigma)^2$. In the original formulation (3.1), this gives

$$\begin{aligned} S &= (\sigma)^2 Tr(\mathbf{g}^{-1}\partial_\mu \mathbf{g})^2 + \partial_\mu \sigma^2 - \mu^2/2\sigma^2 + g/(24N)\sigma^4 \\ &= (\sigma_0)^2 Tr(\mathbf{g}^{-1}\partial_\mu \mathbf{g})^2 + \partial_\mu \sigma^2 + (g/(4N)\sigma_0^2 - \mu^2/2)\delta\sigma^2 + \text{interaction} \end{aligned} \tag{3.21}$$

which clearly identifies the $\pi_i, i = 1, ..., N-1$ as the $N-1$ Goldstones. Note that the π_i fields occur in the action only thru' derivatives, which reflects the fact that $\pi_i(x) \to \pi_i(x) + \epsilon_i$ is a symmetry of the action (eps_i is a zero mode). The finite form of this symmetry is

$$\mathbf{g}(x) \to \mathbf{g_1}\ \mathbf{g}(x)$$

- we find the emergence of an NLSM in the broken phase, Eqn, (3.21).
- **Mermin-Wagner theorem**: Note that $\sigma^2_{\text{quantum}} - \sigma^2_{\text{class}}$ (mentioned above, and described in (3.10), (3.19)), which is given by

  $$\int \frac{d^d k}{k^2}$$

  has an IR divergence for $d \leq 2$ (hence it is infinite even in the presence of the uv cut-off Λ). This implies that σ_{quantum} in (3.19) is undefined. One says that IR fluctuations completely destroy the magnetic order in 2 and less dimensions.

  Another way of seeing this is to note

  $$m_c^2 = -\mu_c^2 = -\frac{g}{6}\int_k \frac{1}{k^2}$$

  which diverges for $d \leq 2$: $m_c^2 = \infty$ Hence a phase transition never happens, can't have $m^2 < -\infty$.

- **The magnetization Exponent β**

  At $m^2 \to m_c^2 - 0 = -\mu_c^2 - 0$, $\sigma \to 0+$. we will see below that at this point, $\xi \to \infty$ and the system goes critical $T = T_c$. Thus,

$$\tau = m^2 - m_c^2 \propto |m| - |m_c|, T - T_c \propto \tau \tag{3.22}$$

  measures deviation from criticality.

$$\rho - \rho_c = -6/g\ \tau$$

  Hence

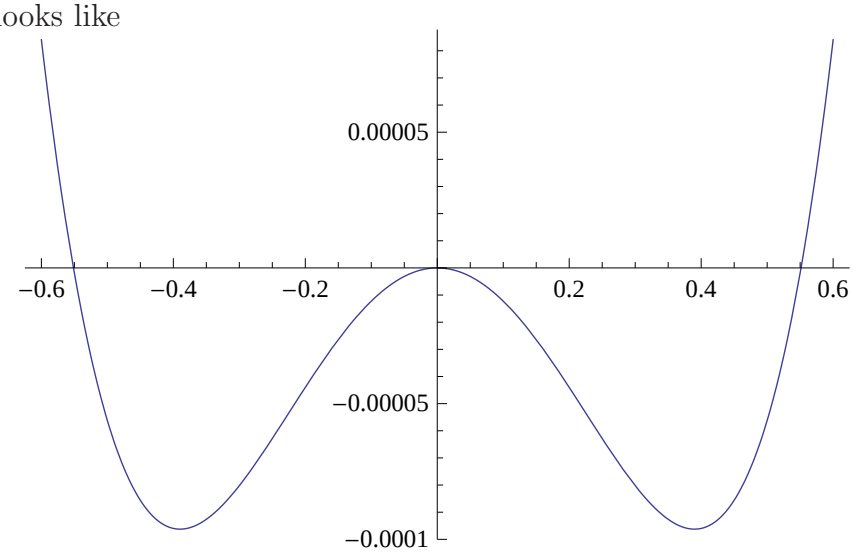
$$\sigma = \sqrt{-6\tau/g} \propto (-\tau)^\beta, \beta = 1/2$$

  which is the mean field exponent. Thus, β agrees with the mean field or quasi-gaussian value in any d ($d > 2$). [β obtainable from power counting].

- In d=3, Eq. (3.14), ignore the $\lambda^{3/2}$ term (since this is subleading in λ/Λ). Integrate out ρ, λ from V_{eff}. We get (ignoring σ-independent terms)

$$V_{eff} = N\frac{\pi^2}{24}\left(g\sigma^4 + 12\sigma^2(\mu_{\text{cr}}^2 - \mu^2)\right) \tag{3.23}$$

  Here we have written $m^2 = -\mu^2$. For $\mu = 2\mu_{\text{cr}}$ (well into the broken phase). For $\Lambda = 1$ (to set the units), and $g = .1$, the potential (3.23) looks like



Symmetric phase

Consider the second solution in (3.18):

$$\sigma = 0, \lambda \neq 0 =: M^2$$

. From (3.12), M^2 is the mass of the order parameter field σ (unlike in footnote 4, in the unbroken phase, we have a non-zero saddle point value of M^2 and the mass corrections coming from $O(1/N)$ can be ignored):

$$\langle \sigma(p)\sigma(-p)\rangle = \frac{1}{p^2 + M^2} \tag{3.24}$$

This gives the disordered (nonmagnetic) phase. In this case, (3.15) becomes

$$\begin{aligned} &\lambda = M^2 = 2U'(\rho) = m^2 + g/6\ \rho \\ &\rho = \int_k \frac{1}{k^2 + M^2} = \rho_c + \int_k (\frac{1}{k^2 + M^2} - \frac{1}{k^2}) = \rho_c + (\Omega_d(M) - \Omega_d(0) \end{aligned} \tag{3.25}$$

where

$$\Omega_d(M) = \int_k \frac{1}{k^2 + M}$$

Note that, using (3.25), the new version of the gap equation (3.16) becomes

$$\rho = \int_k \frac{1}{k^2 + m^2 + g/6\ \rho}$$

and agrees with (3.11). In $d = 3$,

$$\begin{aligned} \Omega_3(M) &= \frac{1}{2\pi^2}(\Lambda - \tan^{-1}(\Lambda/M)) \\ \rho - \rho_c &= -\frac{M}{2\pi^2}\tan^{-1}(\Lambda/M) \end{aligned} \tag{3.26}$$

Criticality

The two phases meet when we demand in (3.25) that $\lambda = 0$. This gives

$$U'(\rho_c) = \frac{m^2}{2} + \frac{g}{12}\rho_c = \frac{m^2}{2} + \frac{g}{12}\int_k 1/k^2 = 0,$$

The system goes critical, since

$$\langle \sigma(p)\sigma(-p)\rangle = \frac{1}{p^2+M^2} \to \frac{1}{p^2} \tag{3.27}$$

Hence $\langle s(0)s(x)\rangle$ has a power-law behaviour, $\xi \to \infty$. Note the absence of any wave-function renormalization [we do not have Z/p^2]. Hence, at the critical point, the anomalous dimension vanishes:

$$\gamma_\sigma = 0 \tag{3.28}$$

Critical region: $M \ll \Lambda$ (as we approach it from the disordered phase).

Exercise 1.: Define $\Delta\rho = (\rho - \rho_c)/\Lambda^{d-2}$. Assume $d > 2$ and Show that in the critical region $M \ll \Lambda$

$$d < 4 : \Delta\rho = [\Omega_d(M) - \Omega_d(0)]/\Lambda^{d-2} = A(M/\Lambda)^{d-2} + ...$$

(see, e.g. (3.26)).

$$d = 4 : \Delta\rho = A(M/\Lambda)^2 \ln(M/\Lambda) + ...$$

$$d > 4 : \Delta\rho = B(M/\Lambda)^2 + ...$$

where ... represent terms which have higher powers of M/Λ.

Exercise 2: Show, using the definition (3.22), that

$$M^2 = \tau + 2U''(\rho_c)(\rho - \rho_c) + O(\rho - \rho_c)^2$$

In $d = 3$, using (3.26) and the above exercise, we get

$$M^2 = \tau - \frac{gM}{12\pi^2}\tan^{-1}(\Lambda/M) = \tau - \frac{g}{24\pi}\left(M - \frac{2M^2}{\pi\Lambda} + O(M^3)\right) \tag{3.29}$$

Sundry exponents

Note that $\tau \propto (T - T_c)$. As we approach criticality, by def

$$\xi \sim |\tau|^{-\nu}, \quad \sigma \sim (-\tau)^\beta, \quad \sigma \sim H^{1/\delta}, \quad \chi \sim |\tau|^{-\gamma}, \quad C_H \sim |\tau|^{-\alpha} \tag{3.30}$$

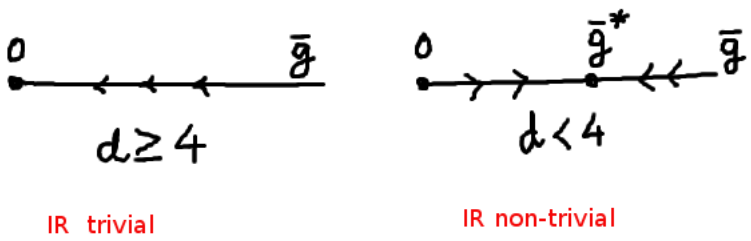
At criticality,

$$\langle \sigma(0)\sigma(x)\rangle = G(x) = \frac{1}{|x|^{d-2+\eta}} \tag{3.31}$$

Clearly $\eta = 2\gamma_\sigma$, where γ_σ is the anomalous dimension of σ. By (3.28), we have $\eta = 0$, a Gaussian (=free field) result in any dimension.

β has already been computed above. We will compute ν, η.

Exercise 3: (Computation of ν)



Combine Ex.1 and Ex.2. Show that

1. $d > 4$: $M^2 \sim \tau$, hence $\xi \sim \tau^{-1/2}, \Rightarrow \nu = 1/2$. [as we would get in mean field theory (i.e. ν has the power counting value)]. This indicates that the $d > 4$ theory has the free field theory as the IR fixed point.

2. $d \in (2, 4)$: $M^{d-2} \sim \tau$, hence $\xi \sim \tau^{-1/(d-2)}$. Hence $\nu = 1/(d-2) = 1/(2-\epsilon), d = 4 - \epsilon$. This indicates the existence of a non-trivial CFT whose ν is showing up here (this is the **Wilson-Fisher f.p.** — see before).

3. $d = 4$: $M^2 \sim \tau/(\ln(\Lambda/M)) \sim \tau/\ln\tau$. Log correction to power law behaviour.

Comparison with ϵ expansion

We found above $\nu^{-1} = (2-\epsilon)(1 + O(1/N)$.

This matches with Peskin (13.54): $\nu^{-1} = 2 - \epsilon(1 + O(1/N) + O(\epsilon^2)$. In fact, large N is a reasonable assumption in our system, we are doing better

than the ϵ expansion; we have essentially proved that the $O(\epsilon)^2$ corrections are actually $O(1/N)$ and vanish at large N, even if $\epsilon = 1$. Peskin's expression would appear to imply a relative error of $O(\epsilon)$ which, for $\epsilon = 1$, is 100% !. On the other hand, for a 3D ferromagnet, $N = 3$, and the relative error is at most $O(1/N) \sim 33\%$.

Table 13.1. Values of Critical Exponents for Three-Dimensional Statistical Systems

Exponent	Landau	QFT	Lattice	Experiment	
$N = 1$ Systems:					
γ	1.0	1.241 (2)	1.239 (3)	1.240 (7)	binary liquid
				1.22 (3)	liquid-gas
				1.24 (2)	β-brass
ν	0.5	0.630 (2)	0.631 (3)	0.625 (5)	binary liquid
				0.65 (2)	β-brass
α	0.0	0.110 (5)	0.103 (6)	0.113 (5)	binary liquid
				0.12 (2)	liquid-gas
β	0.5	0.325 (2)	0.329 (9)	0.325 (5)	binary liquid
				0.34 (1)	liquid-gas
η	0.0	0.032 (3)	0.027(5)	0.016 (7)	binary liquid
				0.04 (2)	β-brass
$N = 2$ Systems:					
γ	1.0	1.316 (3)	1.32 (1)		
ν	0.5	0.670 (3)	0.674 (6)	0.672 (1)	superfluid ^{4}He
α	0.0	−0.007 (6)	0.01 (3)	−0.013 (3)	superfluid ^{4}He
$N = 3$ Systems:					
γ	1.0	1.386 (4)	1.40 (3)	1.40 (3)	EuO, EuS
				1.33 (3)	Ni
				1.40 (3)	$RbMnF_3$
ν	0.5	0.705 (3)	0.711 (8)	0.70 (2)	EuO, EuS
				0.724 (8)	$RbMnF_3$
α	0.0	−0.115 (9)	−0.09 (6)	−0.011 (2)	Ni
β	0.5	0.365 (3)	0.37 (5)	0.37 (2)	EuO, EuS
				0.348 (5)	Ni
				0.316 (8)	$RbMnF_3$
η	0.0	0.033 (4)	0.041 (14)		

The values of critical exponents in the column 'QFT' are obtained by resumming the perturbation series for anomalous dimensions at the Wilson-Fisher fixed point in $O(N)$-symmetric ϕ^4 theory in three dimensions. The values in the column 'Lattice' are based on analysis of high-temperature series expansions for lattice statistical mechanical models. The values in the column 'Experiment' are taken from experiments on critical points in the systems described. In all cases, the numbers in parentheses are the standard errors in the last displayed digits. This table is based on J. C. Le Guillou and J. Zinn-Justin, *Phys. Rev.* **B21**, 3976 (1980), with some values updated from J. Zinn-Justin (1993), Chapter 27. A full set of references for the last two columns can be found in these sources.

4 Large N limit of interacting $U(N)$ fermions

Consider the following problem

$$Z = \int D\bar{\psi} D\psi \exp[-S]$$
$$S = -\int d^d x [\bar{\psi}_i \partial\!\!\!/ \psi_i + g/(2N)(\bar{\psi}_i \psi_i)^2] \tag{4.1}$$

$g > 0$ appears to be the wrong sign, as it would have been for bosons: $\int D\phi \exp[g \int_x \phi^4]$ would be undefined for $g > 0$. However, the sign $g > 0$ is correct, see remarks below (4.5).

We will first consider this in $d = 2$. This is the original Gross-Neveu model, which is a simpler field theory than QCD, yet shares important features with it:

- asymptotic freedom (+ uv completeness) and dimensional transmutation
- dynamical mass generation and $\bar{q}q$ condensate
- dynamical 'chiral' symmetry breaking
- large N limit

Note

$$[\psi] = (d-1)/2, \quad [\psi^4] = 2(d-1), \quad [\int d^d x \ \psi^4] = d - 2$$

Hence the quartic interaction is non-renormalizable in $d > 2$ and renormalizable in $d \leq 2$. In fact, the theory will turn out to be UV complete.

Dirac-o-logy: Gamma matrices:

$$\gamma_1 = \sigma_1, \ \gamma_2 = \sigma_2, \ \gamma_5 = -i\gamma_1\gamma_2 = \sigma_3$$

2-dim Dirac fermion:

$$\psi_i = \begin{pmatrix} \psi_i^1 \\ \psi_i^2 \end{pmatrix}, \quad \bar{\psi}_i = \begin{pmatrix} \bar{\psi}_i^1 & \bar{\psi}_i^2 \end{pmatrix}, \quad i = 1, 2, ...N$$

$SO(1,1)$ in Lorentzian becomes $SO(2)$ in Euclidean. $SO(2)$ transformation of the Dirac fermion:

$$S_{12} = \frac{i}{4}[\gamma_1, \gamma_2] = -\frac{1}{2}\sigma_3$$

$$\Lambda_{1/2} = e^{-i\theta S_{12}} = e^{i\theta/2.\sigma_3} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

$$\begin{pmatrix} \psi_i^1 \\ \psi_i^2 \end{pmatrix}, \quad \to \begin{pmatrix} \psi_i^1 e^{i\theta/2} \\ \psi_i^2 e^{-i\theta/2} \end{pmatrix},$$

Hence $\psi^1 = \psi^L$ (anticlockwise), $\psi^2 = \psi^R$ (clockwise). Similarly,

$$\begin{pmatrix} \bar{\psi}_i^1 & \bar{\psi}_i^2 \end{pmatrix} \to \begin{pmatrix} \bar{\psi}_i^1 e^{-i\theta/2} & \bar{\psi}_i^2 e^{i\theta/2} \end{pmatrix}$$

In other words, $\bar{\psi}$ transform in the complext conj. repn. $\Lambda^*_{1/2}$. note that $\bar{\psi}^1 = \bar{\psi}^R$ (clockwise), $\bar{\psi}^2 = \bar{\psi}^L$ (anticlockwise).

Kinetic: $\bar{\psi}_L \partial\!\!\!/ \psi_L$ or $\bar{\psi}_R \partial\!\!\!/ \psi_R$. (see Feynman diagram below). Hence we have only LL or RR propagators, as expected in a massless theory.

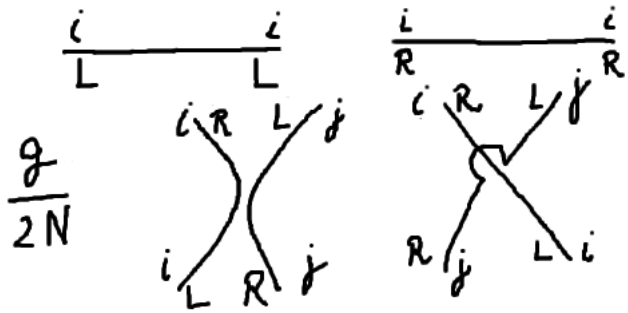
Symmetries:

- $U(N)$ symmetry: $\psi_i \to U_i^j \psi_j$, $\bar{\psi}_j \to U^{-1}{}_j^i \bar{\psi}_i$, $U^{-1} = U^\dagger$ (no $U(N)_L \times U(N)_R$ symmetry since the quartic term has the structure $(\bar{L}R + \bar{R}L)^2$, which contains the term $\bar{L}R\ \bar{L}R$ which preserves only the diagonal $U(N)$).

- Discrete (Z_2) chiral symmetry:

$$\begin{aligned} &\psi_i \to \gamma_5 \psi_i,\ \bar{\psi}_i \to -\bar{\psi}_i \gamma_5 \\ &\text{Or, } \psi_i^{L,R} \to \pm\psi_i^{L,R},\ \bar{\psi}_i^{L,R} \to \pm\bar{\psi}_i^{L,R} \end{aligned} \tag{4.2}$$

  This prevents the generation of any mass term in perturbation theory, since a mass term is of the kind $\bar{R}L$, which picks up a minus sign under the above Z_2 (see a diagrammatic proof below). The large N method allows us to prove a nonperturbative breaking of this symmetry and generation of a mass term.

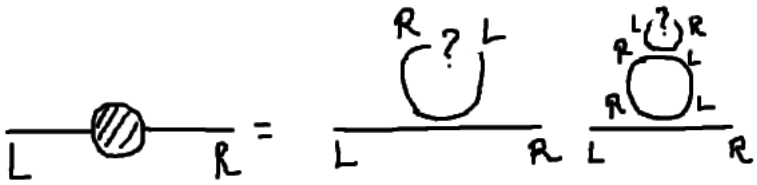


Propagators and vertices

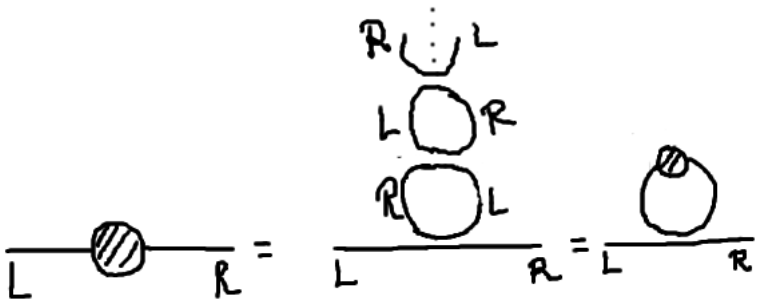
Propagators:

$$\langle \bar{\psi}(p)\psi(-p)\rangle = \frac{-i}{\not{p}} = \frac{-i\not{p}}{p^2} = -\langle \psi(p)\bar{\psi}(-p)\rangle$$

Diagrammatic Proof that No mass can be generated in any finite order of perturbation theory.



In can only be generated in an infinite order of perturbation theory:



where we have kept only the leading large N diagrams.

Exercise 1 From the above diagrammatic equation, derive the 'gap' equation (similarly to the bosonic case)

$$\delta m = \frac{\delta m \; g}{\pi} \int_0^\Lambda \frac{p \; dp}{p^2 + \delta m^2} \tag{4.3}$$

Exercise 2 Self-consistent method: Assume

$$\langle \bar{\psi}_i \psi_i \rangle = N\sigma \tag{4.4}$$

This gives rise to an effective mass δm of the fermion. Using this, reevaluate the LHS of the above equation and equate it to the RHS. Show that you get the same equation as (4.3).

4.1 Functional Methods

Gaussian trick: Consider the identity

$$\int D\sigma \; \exp[-\int_x \left(N\sigma^2/(2g) - \sigma\bar{\psi}_i\psi_i\right)] = (\;) \exp[\int_x \frac{g}{2N}(\bar{\psi}_i\psi_i)^2] \tag{4.5}$$

Thus,

$$Z = \int D\bar{\psi} D\psi D\sigma \ \exp[-S]$$
$$S = \int d^d x [N\sigma^2/(2g) - \bar{\psi}_i(\partial\!\!\!/ + \sigma)\psi_i] \tag{4.6}$$

The chiral symmetry is manifest in this model as

$$\psi_i \to \gamma_5 \psi_i, \ \bar{\psi}_i \to -\bar{\psi}_i \gamma_5, \ \sigma \to -\sigma \tag{4.7}$$

Note the Dyson-Schwinger equation (Quantum Eq. of motion):

$$\langle (N\sigma/g - \bar{\psi}_i \psi_i)_x .... \rangle = 0 \tag{4.8}$$

This is derived from

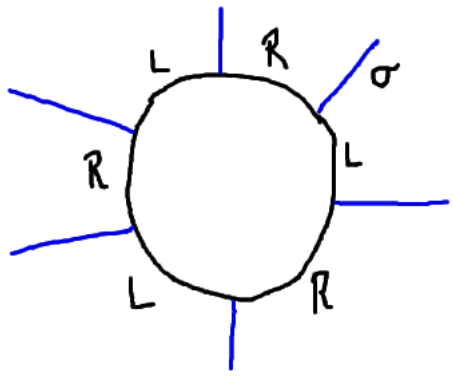
$$0 = \int D\bar{\psi} D\psi D\sigma \ \frac{\delta}{\delta \sigma_x} (\exp[-S])$$

Starting from (4.6) if we integrate σ we get the original theory (4.1) (thus undoing the Gaussian trick), but however since (4.6) is quadratic in the fermions, we can integrate out the fermions, yielding (using (A.2))

$$Z = \int D\bar{\psi} D\psi D\sigma \ \exp[-S]$$
$$S = \int d^d x [N\sigma^2/(2g) - N \mathrm{Tr} \ \ln \ (\partial\!\!\!/ + \sigma)] = N \int d^d x V[\sigma] \tag{4.9}$$

Note the importance of the auxiliary field being a singlet! which causes an overall factor of N in front.

The 'Tr ln' represents the following diagram:



Note the even number of σ's, this reflects the Z_2 symmetry of (4.7). The dynamically generated solution (4.14) spontaneously breaks this symmetry, nonperturbatively.

Large N limit

In the limit $N \to \infty, g =$ fixed ('tHooft limit), we can apply the large N saddle-point method again, as in the bosonic case discussed earlier.

What's the point of solving the σ-theory? It is that by (4.8), we can derive a relation for n-point functions:

$$\langle \frac{g}{N}\bar{\psi}_i\psi_i(x)\ \frac{g}{N}\bar{\psi}_i\psi_i(y)\ ...\rangle = \langle \sigma(x)\ \sigma(y)\ ...\rangle \tag{4.10}$$

The LHS can be evaluated purely in the ψ-theory, whereas RHS can be evaluated purely in the σ-theory, which, therefore, evaluates all correlations of singlets. We can think of the $U(N)$ as 'colour' and the singlets as 'mesons' $(\bar{q}_i q_i)$; indeed if the U(N) is gauged, the individual quarks are not gauge invariant, only the 'mesons' are. Thus, (4.9) in a way, bosonizes the fermions!

4.2 Spacetime independent saddle point

Let us look for a spacetime independent state in the large N limit.

$$V(\sigma) = N[\sigma^2/(2g) - \mathrm{tr}_D \int^{\Lambda} \frac{d^2k}{4\pi^2} \ln(i\not{k} + \sigma)] \tag{4.11}$$

As before, this can be explicitly derived. First, let us look at the EOM:

$$\frac{\sigma}{g} = \frac{\sigma}{\pi} \int_0^{\Lambda} \frac{kdk}{k^2 + \sigma^2} = \frac{\sigma}{2\pi} \ln(1 + \Lambda^2/\sigma^2) \tag{4.12}$$

Exercise Derive this equation. Note that this with (4.3) with the identification of σ^2 with δm^2.

This has 2 solutions for σ:

$$\begin{aligned} &\sigma = 0, \text{ or} \\ &\frac{1}{g} = \frac{1}{2\pi} \ln(1 + \Lambda^2/\sigma^2) \end{aligned} \tag{4.13}$$

For $\sigma \ll \Lambda$, the latter can be solved for σ easily, as

$$|\sigma| = \Lambda \exp[-\frac{\pi}{g}] \tag{4.14}$$

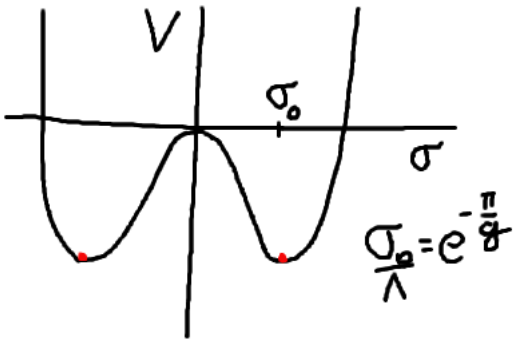
Symmetry breaking

The solution above breaks the discrete chiral symmetry of the original fermions, or of the hybrid (meson-quark) model, (4.7).

The solution gives $|\sigma|$; The precise sign can be chosen by coupling the model to some magnetic field $-h\ s, h \to 0+$, which would force the positive sign of s. Note that the 'phase' of σ is only ± 1. We can's have a $U(1)$ or non-abelian phase because of Mermin-Wagner theorem.

Potential

$V(\sigma)$ can be computed in a simple way. Regard the EOM (4.12) as proving $\partial V/\partial\sigma$. Integrate this equation to get V. It looks like



There is no parameter that can be tuned to restore the symemtry. The model is in only the broken phase, always.

The beta-function

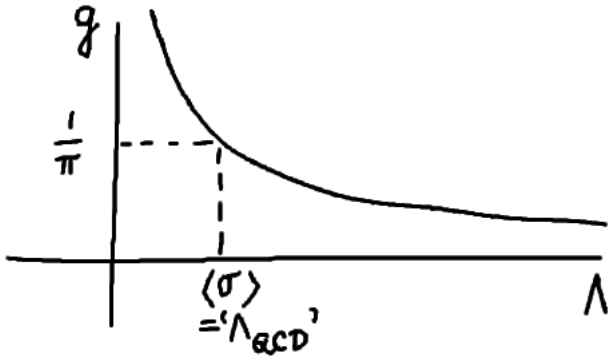
(4.14) or (4.13) trivially gives us the beta-function. note that $\langle\langle\sigma\rangle$ must be RG invariant (note the absence of wave-function renormalization for σ at leading order in large N: it is O(1), see later). Rewrite (4.13) as

$$g = \frac{2\pi}{1 + \ln(\Lambda^2/\sigma^2)} \tag{4.15}$$

Since σ must be kept independent of Λ, we must assign a Λ-dependence to g just as the above equation demands, with σ held constant:

$$\beta_g = \frac{dg}{d\ \ln\Lambda} = -\frac{4\pi}{(1 + \ln(\Lambda^2/\sigma^2))^2} = -\frac{1}{\pi}g^2 \tag{4.16}$$

Asymptotic freedom: Note the minus sign, the same as in QCD. The β-function is exact at large N; therefore there is no other fixed point than $g = 0$, which is a UV attractor. The theory is clearly sensible at high energies. In the IR, it grows.



Chiral Symmetry breaking and dynamical mass generation

In (4.6) we now have a term $\langle\sigma\rangle\bar{\psi}_i\psi_i$, which gives a mass to the fermion and breaks the chiral Z_2 symmetry dynamically.

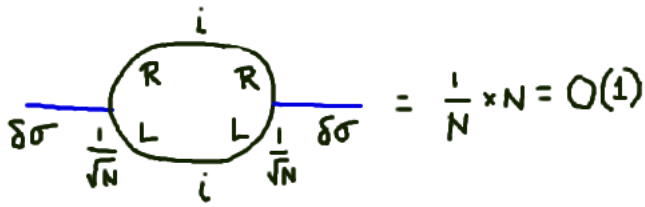
1/N expansion

At the leading order, σ is exactly the classical value (4.14), which we now call σ_{cl}. As we argued before, we can write

$$\sigma = \sigma_{cl} + \frac{1}{\sqrt{N}}\delta\sigma \tag{4.17}$$

Remarks:

(1) The propagator correction is subleading, O(1)



here we have used a vertex $\frac{1}{\sqrt{N}}\delta\sigma\bar{\psi}_i\psi_i$ which follows from (4.6).

(2) A kinetic energy of the 'meson' field [5] gets generated at the subleading order. At leading order, σ does not have a kinetic energy.

(3) **Large N factorization**

Eq. (4.10) becomes

$$\begin{aligned}
&\langle \frac{g}{N}\bar{\psi}_i\psi_i(x)\ \frac{g}{N}\bar{\psi}_i\psi_i(y)\ ...\rangle = \langle(\sigma_{cl}(x) + \frac{1}{\sqrt{N}}\delta\sigma(y))\ (\sigma_{cl}(y) + \frac{1}{\sqrt{N}}\delta\sigma(y)...\rangle \\
&= \sigma_{cl}(x)\sigma_{cl}(y)... + O(1/N) \times \text{connected} \\
&= \langle \frac{g}{N}\bar{\psi}_i\psi_i(x)\rangle\ \langle \frac{g}{N}\bar{\psi}_i\psi_i(y)\ ...\rangle + O(1/N) \times \text{connected} \qquad (4.18)
\end{aligned}$$

Exercise Show that the propagator in the above diagram can be obtained as a Taylor expansion of $S[\sigma]$ in (4.9) using (4.17). We can see it digrammatically from the Feynman diagram below (4.9) using only two σ-lines, and using (4.17) on those lines (ignore σ_{cl} and keep only $\delta\sigma$).

[5] This 'meson', because of its Yukawa coupling to the 'quarks' in (4.6), plays the same role as a 'Higgs' and gives a mass to the fermions. This is Nambu's idea of $\bar{t}\ t$ as a Higgs. That did not work. However, similar things happen in technicolour theories.

4.3 Dynamical Higgs mechanism in a modified GN model

Consider the following modified GN model

$$
\begin{aligned}
Z &= \int D\bar{\psi} D\psi \exp[-S] \\
S &= -\int d^d x[\bar{\psi}_i \partial\!\!\!/ \psi_i + g/(2N)\{(\bar{\psi}_i \psi_i)^2 - (\bar{\psi}_i \gamma_5 \psi_i)^2] \qquad (4.19)
\end{aligned}
$$

Chiral U(1) Symmetry

The interaction term has the structure $\bar{R}L\bar{L}R$ and hence is invariant under a $U(1) \times U(1)$ symemtry:

$$L \to \exp[i\alpha_L]L, \;\; R \to \exp[i\alpha_R]R$$

This is often represented as

vector symemtry: $(L, R) \to \exp[i\alpha](L, R)$, and

axial symemtry: $(L, R) \to (\exp[i\alpha]L, \exp[-i\alpha]R)$, or alternatively

$$\psi \to \exp[i\alpha\gamma_5]\psi \qquad (4.20)$$

In the original NJ model, this was not present because of $(\bar{R}L)^2$ terms.

Gaussian auxiliary variable

$$
\begin{aligned}
Z &= \int D\bar{\psi} D\psi D\sigma_1 D\sigma_1^* \;\; \exp[-S] \\
S &= \int d^d x[N|\sigma_1|^2/(2g) - \bar{\psi}_i(\partial\!\!\!/ + \sigma_1(1-\gamma_5) + \sigma_1^*(1+\gamma_5))\psi_i] \\
S &= \int d^d x[N\sigma^2/(2g) - \bar{\psi}_i(\partial\!\!\!/ + \sigma \; e^{i\phi}(1-\gamma_5) + \sigma \; e^{-i\phi}(1+\gamma_5))\psi_i] \qquad (4.21)
\end{aligned}
$$

In this representation, the axial symmetry (4.20) is again preserved, provided the fermion transformation is augmented as

$$\psi \to \exp[i\alpha\gamma_5]\psi, \;\; \sigma \to \sigma, \;\; \phi \to \phi - 2\alpha \qquad (4.22)$$

After integrating the fermions out, we get

$$S[\sigma,\phi] = N\int d^dx[\sigma^2/(2g) - \mathrm{Tr}\ln(\partial\!\!\!/ + \sigma\ e^{i\phi}(1-\gamma_5) + \sigma\ e^{-i\phi}(1+\gamma_5))] \tag{4.23}$$

In this formulation, the symmetry (4.22) now reduces to just

$$\phi \to \phi - 2\alpha \tag{4.24}$$

Goldstone: By using the above symmetry, we can see that the constant mode of ϕ decouples from the theory. Hence S must only contain derivatives of ϕ; this in particular precludes $m^2\phi^2$ kind of terms. Hence ϕ is a Goldstone. In $d = 2$, this conclusion will be destroyed by IR divergences from $O(1/N)$ terms.But in $d = 2+\epsilon$ those will be absent and ϕ will be a genuine Goldstone.

Large N

As before, we look for spacetime independent saddle points. In that case, using (4.24) we see that the action is independent of ϕ, so we can put $\phi = 0$. The resulting S gives the same effective potential as before, (4.11). Thus, we have proved that $\langle\sigma\rangle \neq 0$ as before. The choice $\phi = 0$ is arbitrary; by choosing a suitable 'magnetic field' we can make ϕ point in any direction. Thus, the axial symmetry is broken in $d = 2+\epsilon$ dimensions. (see above for remarks on Goldstone).

Gauging the axial U(1) symmetry

$$\begin{aligned}
Z &= \int D\bar\psi D\psi \exp[-S] \\
S &= -\int d^dx[\bar\psi_i D\!\!\!/\psi_i + \frac{1}{4e^2}F_{\mu\nu}^2 + g/(2N)\{(\bar\psi_i\psi_i)^2 - (\bar\psi_i\gamma_5\psi_i)^2] \\
D_\mu &= \partial_\mu + iA_\mu
\end{aligned} \tag{4.25}$$

Eq. (4.21) now changes to

$$\begin{aligned}
Z &= \int D\bar\psi D\psi D\sigma D\phi DA_\mu\ \exp[-S] \\
S &= \int d^dx[N\sigma^2/(2g) - \bar\psi_i(D\!\!\!/ + \frac{1}{4e^2}F_{\mu\nu}^2 + \sigma\ e^{i\phi}(1-\gamma_5) + \sigma\ e^{-i\phi}(1+\gamma_5))\psi_i]
\end{aligned} \tag{4.26}$$

After integrating the fermions out, we get

$$S[\sigma,\phi,A] = N\int d^dx[\sigma^2/(2g) - \mathrm{Tr}\ln(\gamma_\mu(\partial_\mu + A_\mu)+$$
$$\sigma\ e^{i\phi}(1-\gamma_5) + \sigma\ e^{-i\phi}(1+\gamma_5)) + \frac{1}{4e^2}F_{\mu\nu}^2] \tag{4.27}$$

Gauge symmetry

The symmetry (4.24) is now local, with the additional gauge transformation of A_μ:

$$\phi \to \phi' = \phi - 2\alpha, A_\mu \to A'_\mu = A_\mu - \partial_\mu\alpha \tag{4.28}$$

Using this, we can now get rid of ϕ entirely (not just the constant mode) by fixing a gauge

$$\phi' = 0$$

We can do this by choosing

$$\alpha = \phi/2$$

which gives a gauge field

$$A'_\mu = A_\mu - \frac{1}{2}\partial_\mu\phi \tag{4.29}$$

In this gauge we have

$$S[\sigma,A'] = N\int d^dx[\sigma^2/(2g) - \mathrm{Tr}\ln(\gamma_\mu(\partial_\mu + A'_\mu) + \sigma) + \frac{1}{4e^2}F_{\mu\nu}^2] \tag{4.30}$$

This action has a $\sigma^2 {A'_\mu}^2$ term. To see this, it is enough to concentrate on spacetime-independent fields, including A'_μ. Schematically, the new gap equation becomes

$$\sigma/g - \sigma\int\frac{d^dk}{(k+A')^2+\sigma^2} = 0$$

Integrating this w.r.t σ to get $V(\sigma,A')$, as we had done before, and expanding in A', we find that V contains a term $\sigma^2 A'^2$ At the large N saddle point, the effective action (4.27), therefore must have a term

$$\sigma^2 A'^2 = \left(\sigma_{cl} + \frac{1}{\sqrt{N}}\delta\sigma\right)^2 A'^2 = \sigma_{cl}^2 A'^2 + ...,\ m_A = \sigma_{cal} \tag{4.31}$$

This shows the appearance of a mass term for the gauge field. [6] [This is in the gauge $(\phi' = 0, A')$; in terms of (ϕ, A) we will have $m_A^2(A_\mu - \frac{1}{2}\partial_\mu\phi)^2$].

Exercise: Complete this argument.

This is a dynamical Higgs mechanism, in which the Higgs field $\sigma \exp[i\phi]$ is emergent.

4.4 QCD and Nambu Jona-Lasinio (NJL)

QCD is described by

$$S = \int d^4x \left(-\frac{1}{4}(F^i_{\mu\nu,j})^2 - \bar{\psi}_{ia}(\not{D}^i_j - m_a\delta^i_j)\psi^{ja} \right) \tag{4.32}$$

where $i = 1, ..., N$ reprsents colours, and $a = 1, 2, ..., N_f$ represents flavours. In real-life QCD, $N = 3, N_f = 6$.

Note that Λ_{QCD} (defined in (5.3), (5.4)) is of the order $200 - 300$ MeV. The flavours u, d, s are light compared to this; (m_s is of the order 100 Mev, so it's not very light; the corrections to the scenario below because of non-zero m_s can often be worked out).

Chiral limit

u, d, s are massless, whereas c, t, b are considered infinitely heavy, so that they decouple from the system. In this limit (4.32) becomes

$$S = \int d^4x \left(-\frac{1}{4}(F^i_{\mu\nu,j})^2 - \bar{\psi}^L_{ia}\not{D}^i_j\psi^{ja}_L - \bar{\psi}^R_{ia}\not{D}^i_j\psi^{ja}_R \right) \tag{4.33}$$

where L, R are the left-, right-components of the quarks. We now have $N_f = 3$.

Chiral symmetry:

$$\psi^{ja}_L \to U_{L}{}^j_i\psi^{j,a}, \ \psi^{ja}_R \to U_{R}{}^j_i\psi^{j,a} \tag{4.34}$$

[6] **There is an important subtlety in precisely** $d = 2$, the expansion of $V(\sigma, A)$ to quadratic order gives $m_A^2 A_\mu^2$, $m_A^2 = e^2 N/\pi$ (see Gross-Neveu, remark below Eq. 7.6; recall that in $d = 2$, e^2 has the dimension of mass^2). In $d = 4$, the quadratic term involves a non-universal (cut-off-dependent) piece and a universal piece which is $m_A^2 A_\mu^2$, with $m_A^2 \propto \sigma_{cl}^2$. The universal piece is subtracted in a physical way in Eq. (64) in Dhar et al. The formula $m_A^2 \propto \sigma_{cl}^2$ is also true in Superconductivity; see, e.g. Eq. 38.13 of Abrikosov, Gorkov and Dzyloshinsky.

where $U_{L,R}$ are $U(N_f)$ constant group elements (global symmetry).

We can write this $U(N_f) \times U(N_f)$ as vector and axial $U(N_f)$:

Vector:

$$\psi_L^{ja} \to U_i^j \psi^{j,a}, \ \psi_R^{ja} \to U_i^j \psi^{j,a}$$

Axial:

$$\psi_L^{ja} \to U_i^j \psi^{j,a}, \ \psi_R^{ja} \to U^{\dagger j}{}_i \psi^{j,a} \tag{4.35}$$

Out of this the axial $U(1)$ has an anomaly. The remaining axial $SU(N_f)$ is anomaly-free, but it gets dynamically broken in the QCD vacuum, as we will see.

It is known experimentally, that in the QCD vacuum

$$\langle \bar{u}\ u \rangle \sim \langle \bar{d}\ d \rangle \ \sim (250 Mev)^3$$

which also signals χSB.

We will explore these using the NJL model.

NJL model

We will assume that the gluon sector is gapped (glueball mass of order Λ_{QCD}) and it has been integrated out. The result will be a Lagrangian which will be local at distances large compared to $1/\Lambda_{QCD}$. This gives

$$S = -\int d^4x (\bar{\psi}^L_{ia} \not{D}^i_j \psi^{ja}_L + \bar{\psi}^R_{ia} \not{D}^i_j \psi^{ja}_R$$

$$+ \frac{g_1}{N} (\bar{\psi}^L_{ia} \psi^{ja}_R)(\bar{\psi}^R_{ja} \psi^{ia}_L) + \frac{g_2}{N} (\bar{\psi}^L_{ia} \gamma_\mu \psi^{ja}_R)(\bar{\psi}^R_{ja} \gamma^\mu \psi^{ia}_L) + \text{(axial U(1) anomaly term)...})$$

$$\text{(axial U(1) anomaly term)} = |\ln \det(\bar{\psi}_R \psi_L)|^2 \tag{4.36}$$

We have only admitted terms consistent with axial $SU(N_f)$ symmetry in (4.35), and admitted a term which represents the $U(1)$ axial anomaly. Since the axial $SU(N_f)$ is anomaly free the Lagrangian will always have this symemtry; the only way it can break is SSB.

Solving NJL at large N (Dhar et al)

We will, for simplicity, ignore the U(1) anomaly term. [See the original papers for its treatment].

Gaussian auxiliary variables

[TO BE TYPED]

4.5 BCS

[NOT DISCUSSED IN DETAIL].

5 Yang-Mills gauge theories

We consider (4.32) again.

Dimensional transmutation

In the limit of massless quarks the theory is given by only one, dimensionless, constant g.

Recall the Callan-Symanzik equation (similar to (3.5), but in stead of considering dependence on the cut-off we now write the dependence on the reference scale M)

$$\left(M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}-n\gamma\right)\Gamma^{(n)}(p_1,...,p_n;M,g)=0,\ \text{Hence} \tag{5.1}$$

Theorem: The solution of the above equation can be written in the form:

$$\Gamma^{(n)}(p_1,...,p_n;M,g)=\Lambda_{QCD}^{\alpha}\ f_n(p_1/\Lambda_{QCD},...p_n/\Lambda_{QCD}) \tag{5.2}$$

where $\Lambda_{QCD}=\Lambda_{QCD}(g,M)$ is defined by the following equation

$$(M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial g})\Lambda_{QCD}(M,g)=0 \tag{5.3}$$

Here $\alpha=[\Gamma^n]$, the mass dimension.

Proof: Assume $\gamma=0$ for simplicity. Then, clearly

$$\Gamma^{(n)}(p_1,...,p_n;M,g)=F(p_1,...,p_n;\Lambda_{QCD}(M,g))$$

which can be proved simply by applying the Callan-Symanzik eqn. without γ. If $[G^{(n)}]=\alpha$, we can take out Λ_{QCD}^{α} from $\Gamma^{(n)}$, leaving the form (5.2). [Proved]

Exercise 1: Extend the proof with $\gamma\neq 0$.

Exercise 2: Generalize (5.2) in the presence of quark masses.

Exercise 3: Show that for $\beta(g)=-b/3\ g^3$, as is the case for YM,

$$\Lambda_{QCD}=M\exp[-1/(bg^2)] \tag{5.4}$$

5.1 Large N gauge theory

The import of the above theorem is that $\Gamma^{(n)}$ in massless QCD for $N = 3$ has no parameters! It's a zero parameter theory. 'tHooft suggested taking a sequence of $SU(N)$ theories and studying a large N limit, which might be simple.

How does it happen in practice?

$$-\Gamma^{(4)}(P, P, P, P; M; g) = \frac{\bar{\lambda}(P)}{N}$$

where

$$\bar{\lambda}(P) = \frac{(4\pi)^2}{b_0 - f_0 \frac{N_f}{N}} \ln(\frac{P^2}{\Lambda^2_{QCD}}) \left[1 - 2 \frac{b_1^0 + \frac{b_1^1}{N^2} - f_1 (\frac{N_f}{N})^2}{(b_0 - f_0 \frac{N_f}{N})^2} \frac{\ln\ln(\frac{P^2}{\Lambda^2_{QCD}})}{\ln(\frac{P^2}{\Lambda^2_{QCD}})} \right] \tag{5.5}$$

where (Peskin, Hatsuda)

$$b_0 = 11/3, \;\; f_0 = 2/3$$

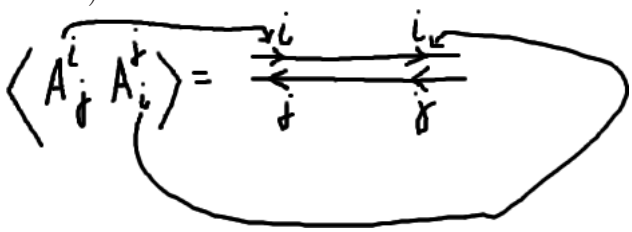
We see an expansion of the form

$$\Gamma^4 = \Gamma^4_0 + O(\frac{1}{N^2}) + O(\frac{N_f}{N}) + ... \tag{5.6}$$

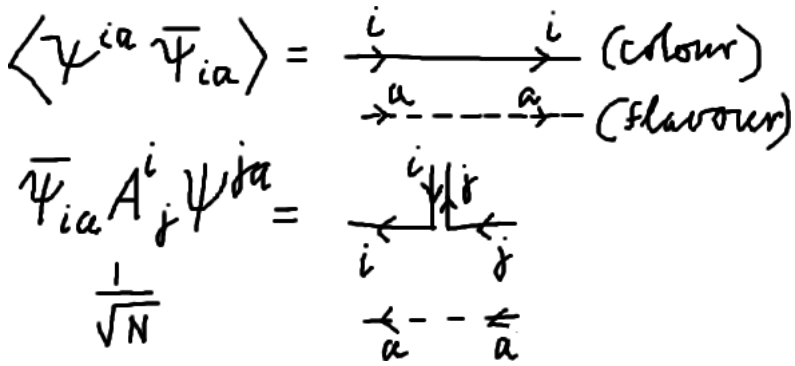
Can we understand this structure without explicit calculation?

Large N Feynman diagrams

Gluon propagator (^{i} = arrow towards the center of diagram, $_i$ = arrow away from the center)



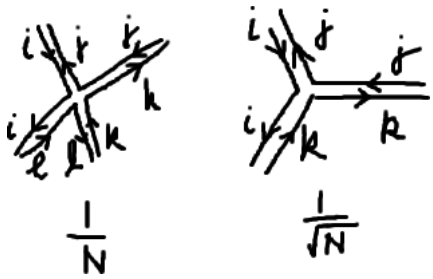
Quark propagator and vertex:



The vertex is $g\bar{\psi}_i A^i_j \psi^j$, where we have defined the 'tHooft coupling

$$g = \sqrt{\frac{\lambda}{N}}. \tag{5.7}$$

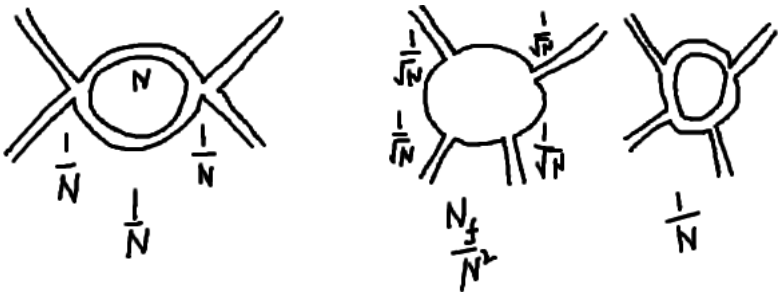
Gauge vertices are the quartic one $g^2 A^i_j A^j_k A^k_l A^l_i = \frac{\lambda}{N} A^i_j A^j_k A^k_l A^l_i$, and the triple vertex (ignoring Lorentz indices): $gA^i_j A^j_k \partial A^k_i = \sqrt{\frac{\lambda}{N}} A^i_j A^j_k \partial A^k_i$.

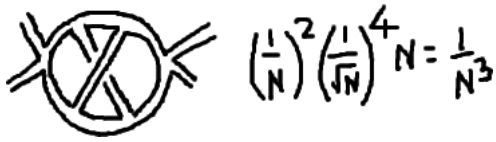


We will find that the large N limit exists, with

$$N \to \infty, \ \lambda = \text{fixed} \tag{5.8}$$

Sample diagrams for $\Gamma^{(4)}$





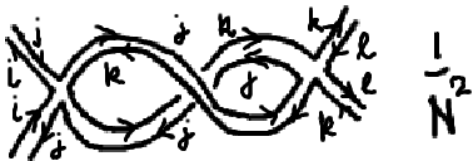
Exercise: Draw all diagrams up to $O(\lambda^3)$. Compare with (5.5) and (5.6).

Lessons:

Each internal quark loop is suppressed by a factor N_f/N. (5.9)

Non-planar diagrams are suppressed; each "bridge" is suppressed by a factor of $1/N^2$. (5.10)

Teaser: the following has a fake "bridge"; it can be twisted away and hence it has the same N-counting as the interaction vertex $\frac{1}{N^2}(TrA^2)^2$.

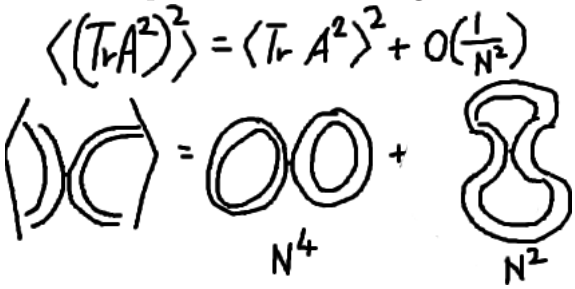


Note that the polarization structure of the external legs here is such that such a $\Gamma^{(4)}$ needs the double trace interaction $\frac{1}{N^2}(TrA^2)^2$. A double trace has 2 traces and hence needs a coefficient $\frac{1}{N^2}$.

Comments:

At leading N, we discard non-planar diagrams. However, there are an infinite number of planar diagrams :(-

How does one solve the problem of summing over these?



Clue: Large N factorization (see figure) suggests the existence of a classical theory! **cf. (GN model, NJL model etc; AdS/CFT)**

5.2 Qualitative results based on Large N Feynman diagrams

Hadron phenomenology: There are aspects of hadron phenomenology which do not have a theoretical explanation (other than by the large N method)

1. Suppression of $q\bar{q}$ sea in hadronic physics; mesons are approximately pure $\bar{q}q$ states; the absence (or at least suppression) of $\bar{q}q\bar{q}q$ exotics.

2. Zweig's rule; the fact that the mesons come in nonets of flavour SU(3); the decoupling of glue states.

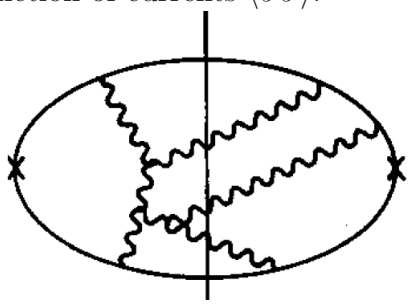
3. The fact that multiparticle decays of unstable mesons are dominated by 2-body meson states, when these are available.

4. Regge phenomenology; the success of a phenomenology that describes strong interactions in terms of tree diagrams with exchange of physical hadraons.

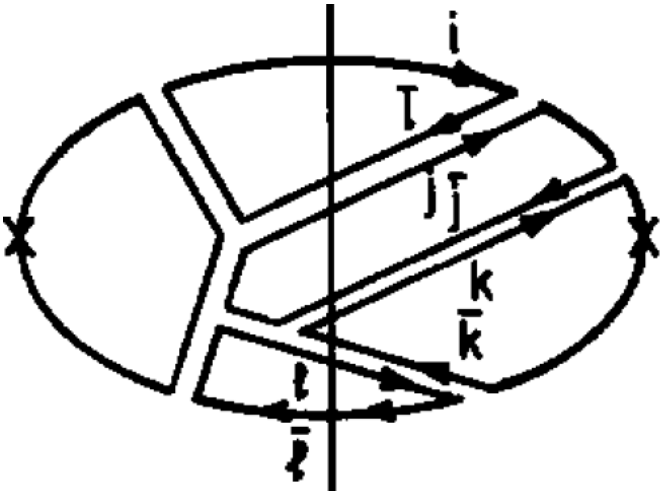
Let us see how to understand these, in turn.

Analysis of the two-point function of currents $\langle JJ \rangle$

Consider a 2-pt function of currents $\langle JJ \rangle$.



which, in double line notation, is



Note the the cut represents an intermediate state

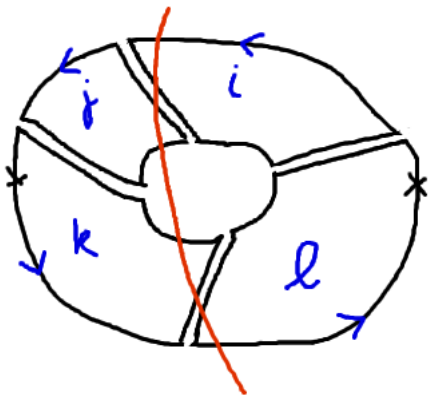
$$\bar{q}_l A^l_k A^k_j A^j_i q^i$$

which is a single hadron, a meson. We can't split this state as a 2-particle state like

$$(\bar{q}_l A^l_k)(A^k_j A^j_i q^i),\ \textit{or}\ [\bar{q}_l (A^l_k A^k_j) A^j_i q^i],\ \textit{or}\ [\bar{q}_l (A^l_k A^k_j A^j_i) q^i]\ \textit{etc.}$$

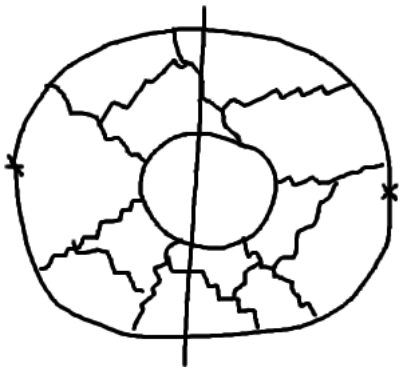
The first option is not there in a confined theory any way. In the second, $(A^l_k A^k_j)$ contains a singlet, but that is 1 in N^2, hence subdominant. Etc.

A similar diagram with an internal quark loop, is



In this diagram, the cut can be interpreted as intermediate two meson states $\bar{q}_i A^i_j q^j$ and $\bar{q}_k A^k_l q^l$.

Exercise: Draw another diagram which shows two intermediate mesons. This has to necessarily contain an internal quark loop. One way is to show it for the diagram below



Similarly, we can show that MG intermediate states (G=glueball) are suppressed.

Exercise: Draw a non-planar diagram which shows an intermediate meson-glueball state. E.g. $(\bar{q}_i A^i_j q^j\, A^l_m A^m_l)$.

This shows that in any leading diagram of $\langle JJ \rangle$, there is only one intermediate gauge-invariant (mesonic) state. Hence it must have a representation:

$$\langle J(k)J(-k)\rangle = \sum_n \frac{a_n}{k^2 - m_n^2}, \;\; a_n = |\langle 0|J|M_n\rangle|^2,$$
$$m_n = \text{meson masses}, \;\; M_n = 1 - \text{meson states} \tag{5.11}$$

There are subleading terms:

$$f(k^2 - (m_1 + m_2)^2)$$

where f has a branch cut at $k^2 - (m_1 + m_2)^2$.

Observations:

1. LHS = O(N). Hence a large N limit

$$\text{Limit}_{N\to\infty} \sum_n \frac{a_n/N}{k^2 - m_n^2} \tag{5.12}$$

exists (is finite). This requires: m_n^2 must be independent of N and $a_n = O(N)$.

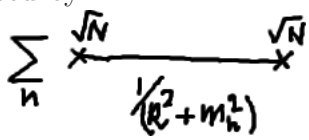
2. Number of mesons must be infinite. If it was finite, with the highest meson mass m_{max}, then for $k \gg m_{max}$, the RHS in (5.11) would have gone as $1/k^2$. However, the LHS, for large k^2 (in the AF region), is known to involve $\log(k^2/\Lambda_{QCD}^2)$.

Exercise: Show the purple part.

3. m_n^2 are real. This follows from the fact that (5.11) represents a Kallen-Lehman spectral representation (sum of delta-functions: $\rho(M^2) = \sum_n \delta(M^2 - m_n^2)$ [Peskin Ch. 7]. **Narrow resonances: width goes as $1/\sqrt{N} \to 0$.**

4. (5.11) and point 1 above show that $|\langle 0|J|M\rangle| \sim O(\sqrt{N})$.

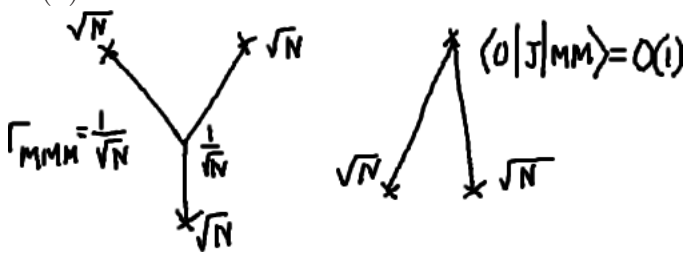
These can be summarised by



Analysis of $\langle JJJ \rangle$

Consider similar fishnets as before, with cuts running in an arbitrary fashion. All these represent 1-particle (1-meson) intermediate states.

These can be represted in two types of diagrams, (i) $\Gamma_{MMM} \sim 1/\sqrt{N}$, (ii) $\langle 0|J|MM\rangle \sim O(1)$.



Thus, we have proved that that the decay of a meson is suppressed by $1/\sqrt{N}$.

Analysis of $\langle JJJJ \rangle$

Back to hadron pheno: explanations

1. Suppression of $q\bar{q}$ sea in hadronic physics: *internal quark loops are suppressed. Internal quark loops represent $q\bar{q}$ popping in and out of vacuum.* mesons are approximately pure $\bar{q}q$ states; the absence (or at least suppression) of $\bar{q}q\bar{q}q$ exotics: *By factorization $\langle(\bar{q}q(x))^2(\bar{q}q(y))^2\rangle = \langle\bar{q}\bar{q}(x)\bar{q}q(y)\rangle^2$. Hence in the putative (meson)^2 state the two mesons do not bind, they propagate independently. This also follows from $\Gamma_{MMMM} \sim 1/N$.*

2. Zweig's rule; the fact that the mesons come in nonets of flavour SU(3):

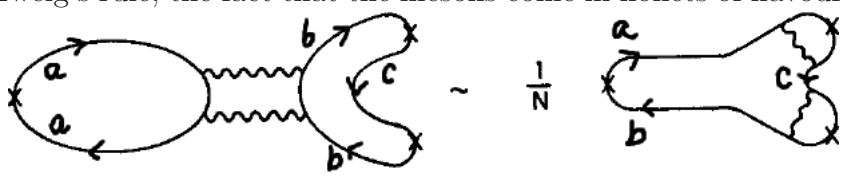


Fig. 31. Zweig's rule.

The nonet structure appears because singlet-octet mixing is suppressed. The figure on the left shows a process involving interactions connecting an octet and a singlet, on the right we have three octets interacting.

the decoupling of glue states: *we saw that $\langle JJ \rangle$ does not have an intermediate glue state.*

3. The fact that multiparticle decays of unstable mesons are dominated by 2-body meson states, when these are available: $\Gamma_{MMM} \sim 1/\sqrt{N}$, $\Gamma_{MMMMM} \sim N^{-3/2}$.

4. Regge phenomenology; the success of a phenomenology that describes strong interactions in terms of tree diagrams with exchange of physical hadraons. *(5.11) shows discrete stable resonances.*

5.3 Baryons

Ref: Witten again.

Bilocal $\sigma(x, y)$

If we have a bi-local four-fermi interaction

$$S_{int} = -\frac{1}{2} \int d^d x \; d^d y \bar{\psi}_i \psi^i(x) V(x, y) \bar{\psi}_j \psi^j(y) \tag{5.13}$$

we should look for a bilocal meson field, equivalent to $\bar{q}\, q$ at separated points $\sigma(x, y) \propto \bar{\psi}_i(x)\psi^i(y)$. In other words, we use the by now familiar trick of using a Gaussian variable $\sigma(x, y)$ so that we express the four-fermi interaction in terms of σ^2 and $\sigma\bar{\psi}\psi$:

$$S_{int} = \int d^d x \; d^d y \left[\frac{N}{2} (\sigma(x, y))^2 - \sigma(x, y) \bar{\psi}_i(x) \psi^i(y) \sqrt{V(x, y) + h.c.} \right] \tag{5.14}$$

By σ-EOM:

$$\sigma(x, y) = \sqrt{V(x, y)} \bar{\psi}_i(x) \psi^i(y)/N \tag{5.15}$$

Bilocal ψ^4 interactions can arise through Coulomb and other interactions. See the BCS discussion in Weinberg II, e.g. Another famous example is 2D QCD ('tHooft), where in the $A_1 = 0$ gauge, $A_0 = -\partial^{-2}(\psi_i^\dagger \psi^i)$. Hence we get (5.13) with $V(x, y) = |x - y|$.

The full action (hybrid) is

$$S[\bar{\psi}, \psi; \sigma(x, y)] = -\int d^d x \; d^d y \; [\bar{\psi}_i(x) \left(\partial\!\!\!/ + m + \sigma(x, y) \sqrt{V(x, y)} + h.c. \right) \psi^i(y) + \frac{N}{2} |\sigma(x, y)|^2] \tag{5.16}$$

After integrating out the fermions, we get

$$S[\sigma(x,y)] = N\left[\frac{1}{2}\int d^dx\ d^dy\ (\sigma(x,y))^2 - Tr\ln\left(\partial\!\!\!/ + m + \sigma\sqrt{V}\right)\right] \quad (5.17)$$

We look for factorizable solutions (quark orbitals)

$$\sigma(x,y) = \sqrt{V(x,y)}\phi^*(x)\phi(y) \quad (5.18)$$

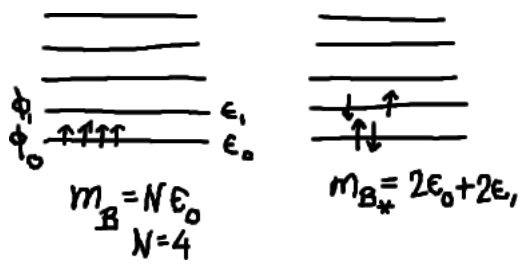
In the limit of heavy mass, the gap equation reduces to (switching to Lorentzian space)

$$\begin{aligned}&[i\partial_t - \partial_x^2/(2m) + V_{MF}[\phi](x)]\phi(x) = 0\\ &V_{MF}(x) = \int d^dy\ \phi^*(y)\phi(y)V(x,y)\end{aligned} \quad (5.19)$$

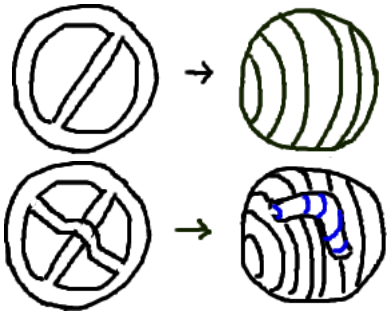
which, of course, is the non-relativistic Hartree equation, familar from atomic physics.

(5.19) has typically an infinite number of solutions, $\phi_0(x), \phi_1(x), ...$ A baryon state (in which all spins are up) will be given by (in its ground state)

$$\Psi(i_1, x_1; ...., i_N, x_N) = \epsilon_{i_1...i_N}\phi_0(x_1)...\phi_0(x_N) \quad (5.20)$$



6 AdS/CFT

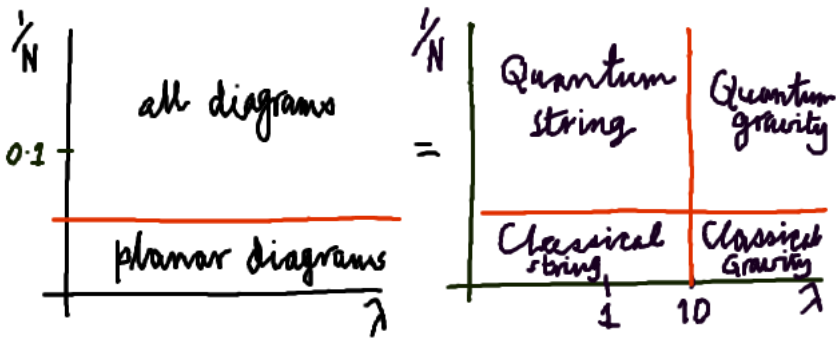


Every time a bridge appears, we have a factor of $1/N^2$. In the string diagram, we have two triple-string vertices for each handle– a factor of g_s^2. Hence $g_s = 1/N$.

'tHooft's genus expansion:

$$\text{vacuum energy} = N^{2-2g} =: (g_s)^{2g-2}$$

In $\mathcal{N}_{SUSY} = 4$ YM theory, the coupling does not run, hence λ is a tunable parameter.



$$G_N \sim 1/N^2, \alpha' \sim 1/(\sqrt{\lambda'})$$

in units $R_{AdS} = 1$. Recall, $G_N = l_s^8 g_s^2$,$\lambda = g_s N, R = l_s \lambda^{1/4}$, implying the above relations (in units R=1, or see p.59 MAGOO review).

Calculations

$$Z_{YM} = \int DA_\mu D\bar{\psi} D\psi D\Phi e^{-S_{YM}} = \int Dg_{\mu\nu} D\}_{\mu\alpha} D\phi ... e^{-\int d^5x \sqrt{g}[R+\Lambda]}|_{\text{AdS}}$$

$$Z_{YM} \langle TrF^2(x)\ TrF^2(y) \rangle = \frac{\partial}{\partial J(x)} \frac{\partial}{\partial J(y)} \exp[-W[J]]$$

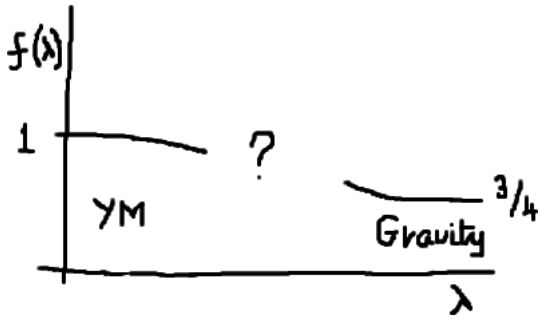
where

$$\exp[-W[J]] = \int Dg_{\mu\nu} D\}_{\mu\alpha} D\phi ...|_{\phi(r=\infty,x)=J(x)} e^{-\int d^5x \sqrt{g}[R+\Lambda]}|_{\text{AdS}}$$

We get

$$F_{YM} = f(\lambda) \frac{\pi^2}{6} N^2 V T^4 + O(N^0)$$

$$F_{gravity} = \bar{f}(\lambda) \frac{\pi^2}{6} N^2 V T^4 + O(N^0)$$



7 Matrix models at large N

A Grassmann path integral

A.1 Coherent states

Let $\{b, b^\dagger\} = 1,\ b^2 = b^{\dagger 2} = 0.$

$$\int d\theta\ \theta = 1,\ |\theta\rangle \equiv \exp[-\theta\ b^\dagger]|0\rangle = |0\rangle - \theta b^\dagger|0\rangle = |0\rangle - |1\rangle,\ \text{check } b|\theta\rangle = \theta|\theta\rangle$$

$$\langle\theta| = \langle 0|\exp[-\bar{\theta} b^\dagger] \overset{\textbf{note}}{=} \langle 0|\exp[b^\dagger\bar{\theta}] = \langle 0| + \langle 1|\bar{\theta} \tag{A.1}$$

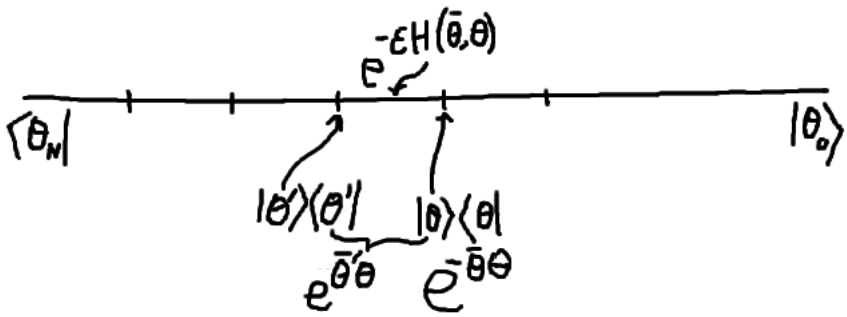
Completeness

$$\mathbf{1} = \int d\bar{\theta} d\theta \exp[-\bar{\theta}\theta]|\theta\rangle\langle\theta|$$

Path integral: use the above resolution of identity at each site of the time lattice. Let $H = b^\dagger b$. We need

$$\langle\theta_{n+1}|1 - \epsilon H(b^\dagger, b)|\theta_n\rangle = (1 - \epsilon H(\bar{\theta}_n, \theta_n)) \exp[\bar{\theta}_{n+1}\theta_n]$$

Building on these sandwiches, we get the full path-integral.



A.2 Gaussian integrals

$$\int d\bar{\theta}\ d\theta e^{-\bar{\theta}\ a\ \theta} = a, \int d\bar{\theta}\ d\theta\ \theta\bar{\theta}\ e^{-\bar{\theta}\ a\ \theta} = 1$$

$$Z \equiv \prod_i \int d\bar{\theta}_i \; d\theta_i e^{\sum_i -\bar{\theta}_i A_{ij} \theta_j} = \prod_i \int d\bar{\theta}_i \; d\theta_i e^{\sum_i -\bar{\theta}_i \; a_i \; \theta_i}$$

$$= \prod_i a_i = \det A = \exp[\mathrm{Tr} \; \ln A] \tag{A.2}$$

$$\prod_i \int d\bar{\theta}_i \; d\theta_i \theta_1 \bar{\theta}_1 \; e^{\sum_i -\bar{\theta}_i A_{ij} \theta_j} = \prod_{i \neq 1} a_i$$

$$\frac{1}{Z} \; \prod_i \int d\bar{\theta}_i \; d\theta_i \theta_1 \bar{\theta}_1 \; e^{\sum_i -\bar{\theta}_i A_{ij} \theta_j} = \frac{1}{a_1} \tag{A.3}$$

This shows the principle of the fermion propagator calculation.

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