General Relativity.

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GR arises by attempting to reconcile Newtonian gravity (experimentally verified) with special relativity (also experimentally verified). However the structure that emerges is not just a generalisation of these two theories, but involves a radically new point of view about spacetime.

Before embarking on this, let us try to get a feeling for curved <u>space</u>. A free non-relativistic particle with coordinates x^i $(i = 1, 2, \dots d)$ has a Lagrangian

$$L = \frac{1}{2} m \dot{x}^2 \qquad \left(\text{here } \dot{x}^2 = \sum_{i=1}^{d} \dot{x}^i \dot{x}^i\right)$$

and equations of motion $\ddot{x}^i = 0$.

Let us try to generalise this Lagrangian, not by adding a potential but by modifying the kinetic term. We keep the fact that it is quadratic in velocities. Then it can be generalised to:

$$L = \frac{1}{2} m \, g_{ij}(x) \, \dot{x}^i \dot{x}^j$$

where $g_{ij}(x)$ is some "sensible" function of the position.

We claim that this describes a particle moving on a space with a non-trivial "metric" $g_{ij}(x)$ which determines the distance between infinitesimally separated points by:

$$ds = \sqrt{g_{ij}(x)\,dx^i dx^j}$$

~~Example~~ Exercise. Consider a particle constrained to move on the surface of a sphere:

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = R^2$$

The free Lagrangian $\frac{1}{2} m \dot{x}^2$ must now be subjected to the constraint above. For this, let us solve the constraint as:

$$x^3 = \sqrt{R^2 - (x^1)^2 - (x^2)^2}$$

Use the notation $(x^1, x^2) = \vec{x}$ or x^a $(a=1,2)$

Then $$\dot{(x^3)} = \frac{1}{2\sqrt{R^2 - \vec{x}^2}} \cdot -2\vec{x}\cdot\dot{\vec{x}}$$

So $$L = \frac{1}{2} m \left(\dot{x}^a \dot{x}^a + \frac{x^a x^b \dot{x}^a \dot{x}^b}{R^2 - x^a x^a} \right)$$

$$= \frac{1}{2} m \left(\cancel{\frac{\dot{x}^b \dot{x}^b (R^2 - x^a x^a) + x^a x^b \dot{x}^a \dot{x}^b}{R^2 - x^a x^a}} \right)$$

$$= \frac{1}{2} m \left(\delta^{ab} + \frac{x^a x^b}{R^2 - x^c x^c} \right) \dot{x}^a \dot{x}^b$$

$$= \frac{1}{2} m \, g_{ab}(x) \, \dot{x}^a \dot{x}^b$$

~~This is the~~ where $g_{ab}(x) = \delta^{ab} + \dfrac{x^a x^b}{R^2 - x^c x^c}$

is the metric of a 2-sphere.

Note that we already know that

$$-R \leq x^a \leq R \qquad , \qquad a = 1, 2$$

(otherwise the $\sqrt{\ }$ above would be imaginary). However in the metric

$$g_{ab}(x) = \delta^{ab} + \frac{x^a x^b}{R^2 - x^c x^c}$$

it is not clear that the x^a must satisfy this constraint. Indeed this should be specified by hand.

However it can be shown that the metric is "badly behaved" for $x^c x^c > R^2$, but this will require more detailed analysis that we will perform later.

Note here that the metric can be made to look different merely by changing coordinates.

Under $x^a = x^a(x')$, we have

$$\dot{x}^a = \frac{dx^a}{dt} = \frac{\partial x^a}{\partial x'^b} \dot{x}'^b$$

~~So~~ $\cancel{L = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m}$

$$L = \frac{1}{2} m \, g_{ab}(x) \, \dot{x}^a \dot{x}^b$$

$$= \frac{1}{2} m \, g_{ab}(x) \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d} \dot{x}'^c \dot{x}'^d$$

Define $g'_{cd}(x') = g_{ab}(x) \dfrac{\partial x^a}{\partial x'^c} \dfrac{\partial x^b}{\partial x'^d}$, then

$$L = \frac{1}{2} m \, g'_{ab}(x') \, \dot{x}'^a \dot{x}'^b$$

We see that the Lagrangian remains form-invariant only if we define g_{ab} not as a simple <u>function</u> but as a <u>tensor</u>:

$g'_{cd}(x')$ is obtained from $g_{ab}(x)$ by multiplying with the extra factors $\dfrac{\partial x^a}{\partial x'^c} \dfrac{\partial x^b}{\partial x'^d}$.

While the form of the Lagrangian then remains unchanged, the functional dependence of the metric <u>does</u> change when we make a coordinate transformation.

It is convenient to work out the <u>infinitesimal</u> <u>form</u> of

$$g'_{cd}(x') = \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d} g_{ab}(x).$$

Suppose $x'^a = x^a - \xi^a$ where ξ^a is small.
We keep only terms of first order in ξ^a.

Now $$g'_{cd}(x') = g'_{cd}(x) - \xi^e \partial_e g_{cd}(x)$$

On the RHS, $$\frac{\partial x^a}{\partial x'^c} = \delta^a{}_c + \partial_c \xi^a$$

So RHS $= (\delta^a{}_c + \partial_c \xi^a)(\delta^b{}_d + \partial_d \xi^b)\, g_{ab}(x)$

$$= g_{cd} + \partial_c \xi^a g_{ad} + \partial_d \xi^b g_{cb}$$

Hence $g'_{cd}(x) - g_{cd}(x) \equiv \delta g_{cd}(x)$

$$= \partial_c \xi^a g_{ad} + \partial_d \xi^b g_{cb} - \xi^a \partial_a g_{cd}$$

In particular if $g_{ab} = \delta_{ab}$ then

$\cancel{\delta g_{ab}}$ $$\boxed{\delta g_{cd} = \partial_c \xi_d + \partial_d \xi_c}$$

<u>Ex</u>: If g_{ab} is not flat, show that we still have

$$\delta g_{cd} = D_c \xi_d + D_d \xi_c$$

where $$D_c \xi_d = \partial_c \xi_d - \Gamma^a_{cd} \xi_a$$

christoffel symbol.

Exercise: Define spherical polar coordinates in the usual way: θ, φ.

Then
$$x^1 = \cancel{R} \cos\theta \cos\varphi$$
$$x^2 = \cancel{R} \cos\theta \sin\varphi$$
$$x^3 = \cancel{R} \sin\theta$$

The constraint $(x^1)^2 + (x^2)^2 + (x^3)^2 = R^2$ is automatically satisfied by fixing $r = R$.
This leaves just two coordinates θ and φ.

Show that $(dx^1)^2 + (dx^2)^2 + (dx^3)^2 = d\theta^2 + \cos^2\theta \, d\varphi^2$

Therefore the 2-sphere Lagrangian can equivalently be written

$$L = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \cos^2\theta \, \dot{\varphi}^2 \right)$$

Thus g_{ab} in these coordinates is

$$\begin{pmatrix} R^2 & 0 \\ 0 & R^2 \cos^2\theta \end{pmatrix} = g'_{ab}(\theta, \varphi)$$

Check that the tensor transformation law correctly reproduces this metric starting from $g_{ab}(x)$ of the previous page; ~~i.e.~~

~~$g'_{ab}(\theta,\varphi) =$ Writing $(z^1, z^2) = (\theta, \varphi)$~~

In other words if $x'^1 = \theta$, $x'^2 = \varphi$ then show that

$$g'_{ab}(x') = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} g_{cd}(x)$$

Question: If someone gave you a Lagrangian

$$L = \frac{1}{2} m \left((\dot{y}^1)^2 + \cos^2 y^1 (\dot{y}^2)^2 \right)$$

would you recognise that (y^1, y^2) are really (θ, φ) ? How ?

This example shows that coordinate transformations change the description but not the "true geometry". The dynamics of a particle on a sphere **obviously** doesn't depend on our using (x^1, x^2) or (θ, φ) !.

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Now let's consider the equation of motion for the Lagrangian

$$L = \frac{1}{2} m \, g_{ij}(x) \, \dot{x}^i \dot{x}^j$$

for an **arbitrary** g_{ij}. The equations are:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0$$

This becomes:

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$$

where $\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(g_{lj,k} + g_{lk,j} - g_{jk,l} \right)$

is called the Christoffel symbol. Here g^{ij} is the matrix inverse of g_{ij}:

$$g^{ij} g_{jk} = \delta^i{}_k .$$

The above equation is called the "geodesic equation" ~~[illegible]~~. It embodies the statement that the particle will choose to extremise its path length. This is no surprise, for the Lagrangian is basically

$$L = \frac{1}{2} m \left(\frac{ds}{dt} \right)^2 \quad \text{where} \quad ds = \sqrt{g_{ij}\, dx^i\, dx^j}$$

is the path length! The equations of motion therefore extremise this quantity.

Exercise: Take the geodesic equation and perform the substitution $x^i = x^i(x')$. Then bring the equation to the form

$$\ddot{x}'^i + \Gamma'^i_{jk}(x')\, \dot{x}'^j \dot{x}'^k = 0$$

What relation do you find between Γ'^i_{jk} and Γ^i_{jk}?