Lecture 3 13/8/09

Notice that special relativity has generated a new equation:

$$\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = 4\pi G_E \vec{J}$$

where $B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$. (This equation was already known!)

~~Now~~ Notice also that a new invariance has arisen. Under $A_\mu \to A_\mu + \partial_\mu \lambda$ where λ is an arbitrary function, $F_{\mu\nu}$ is invariant. Thus our eqn is not only Lorentz invariant but also gauge-invariant.

Now let's try to repeat this trick with Newton's gravitational potential φ_G:

$$\partial_i \partial_i \varphi_G = 4\pi G_N \rho_G$$

This time $\rho_G = \frac{\text{mass}}{\text{Vol}}$ and this is not Lorentz invariant for two reasons. One we already saw, $\frac{1}{\text{Vol}} \sim$ time component of vector. But in relativity, mass $\to$ energy $E = p_0$. So this time the RHS is the time-time component of a 2nd rank tensor! Naming the tensor $T_{\mu\nu}$, we ~~to~~ identify $T_{00} = \rho$, all other components $= 0$ as a special case.

Since the RHS is now ~~4πG~~ $4\pi G_N T_{\mu\nu}$, the LHS must also become a 2nd rank tensor.

Call it $-\frac{\bar{h}_{\mu\nu}}{4}$, so $\bar{h}_{00} = -4\varphi_G$ is a special case (with $h_{0i}, h_{ij} = 0$)

Hence $\Box \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu}$

is the ~~Lorentz~~-invariant extension of Newton's gravity.

Define $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu}$, the trace of $\bar{h}_{\mu\nu}$, and $T = \eta^{\mu\nu}T_{\mu\nu}$ the trace of $T_{\mu\nu}$.

Then $\Box \bar{h} = -16\pi G_N T$

Also note that $T_{\mu\nu}$ has the interpretation of "energy-momentum" with T_{00} = energy density

~~Ex fields~~ like electromagnetic currents, energy-momentum tensors are also conserved:

$$\partial^\mu T_{\mu\nu} = 0$$

as well as symmetric: $T_{\mu\nu} = T_{\nu\mu}$.

Thus we have that $\bar{h}_{\mu\nu}$ is symmetric and

$$\partial^\mu \bar{h}_{\mu\nu} = 0.$$

Now we can rewrite $\Box \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu}$ as

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha\partial_\mu \bar{h}_{\nu\alpha} - \partial^\alpha\partial_\nu \bar{h}_{\mu\alpha} = -16\pi G_N T_{\mu\nu}$$

As with electrodynamics, we are trying to rewrite the equation with some invariance that allows us to "lift" the constraint $\partial^\mu h_{\mu\nu} = 0$. And we have succeeded, because in the above eqn,

$$\partial^\mu \left(\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_\mu \bar{h}_{\nu\alpha} - \partial^\alpha \partial_\nu \bar{h}_{\mu\alpha} \right) = 0$$

is an identity !.

We also see that there is an invariance analogous to gauge invariance:

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial^\alpha \xi_\alpha$$

Check: ~~$\Box \bar{h}_{\mu\nu} \rightarrow \Box \bar{h}_{\mu\nu} +$~~ the variation of $\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_\mu h_{\nu\alpha} - \partial^\alpha \partial_\nu h_{\mu\alpha}$ is:

$$\Box \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial^\alpha \xi_\alpha \right)$$

$$+ \eta_{\mu\nu} \partial^\alpha \partial^\beta \left(\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \eta_{\alpha\beta} \partial^\gamma \xi_\gamma \right)$$

$$- \partial^\alpha \partial_\mu \left(\partial_\nu \xi_\alpha + \partial_\alpha \xi_\nu - \eta_{\alpha\nu} \partial^\gamma \xi_\gamma \right)$$

$$- \partial^\alpha \partial_\nu \left(\partial_\mu \xi_\alpha + \partial_\alpha \xi_\mu - \eta_{\alpha\mu} \partial^\gamma \xi_\gamma \right).$$

$$\begin{aligned}
= \quad & \cancel{\partial_\mu \Box \xi_\nu} + \cancel{\partial_\nu \Box \xi_\mu} - \cancel{\eta_{\mu\nu} \Box \partial \cdot \xi} \\
& + \cancel{2\eta_{\mu\nu} \Box \partial \cdot \xi} - \cancel{\eta_{\mu\nu} \Box \partial \cdot \xi} \\
& - \cancel{\partial_\mu \partial_\nu \partial \cdot \xi} - \cancel{\partial_\mu \Box \xi_\nu} + \cancel{\partial_\mu \partial_\nu \partial \cdot \xi} \\
& - \cancel{\partial_\mu \partial_\nu \partial \cdot \xi} - \cancel{\partial_\nu \Box \xi_\mu} + \cancel{\partial_\mu \partial_\nu \partial \cdot \xi}
\end{aligned}$$

$$= 0 \;!.$$

Though slightly more tedious, the structure so far is quite similar to a generalization of electrodynamics.

The next step is to ~~try~~ and simplify the "gauge symmetry":

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\, \partial \cdot \xi$$

If we define $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$, we find:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu .$$

and $\Box h_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^\alpha\partial_\mu h_{\nu\alpha} - \partial^\alpha\partial_\nu h_{\mu\alpha} - \eta_{\mu\nu}\Box h + \partial_\mu\partial_\nu h = -16\pi G T$

This is simpler, and also we now find a very striking physical interpretation! This is exactly the transformation law for a metric that is infinitesimally close to the identity, under an arbitrary change of coordinates!

Thus combining Newtonian gravity with special relativity has led us to a theory of metrics on <u>spacetime</u>. We must allow spacetime to be "curved". And the gravitational equations must be invariant under <u>general coordinate transformations</u>.

The metric on spacetime cannot be exactly like the familiar Riemannian metric which gives a positive distance between any pair of points. In fact we will see that it is "pseudo-Riemannian".

It is best to start with an example. Special relativity already embodies a metric $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ so the "distance" between two spacetime points is

$$ds^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

which can be positive (spacelike), negative (timelike) or zero (null). So $\eta_{\mu\nu}$ is a pseudo-Riemannian metric.

In going from Galilean to Special Relativity we go from

$$\delta_{ij} \longrightarrow \eta_{\mu\nu}$$

(flat space) (flat spacetime)

Similarly, the calculation we have just performed tells us that we must go from

$$g_{ij} \longrightarrow g_{\mu\nu}$$

(general curved space) (general curved spacetime)

and that if $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ then for small $h_{\mu\nu}$ we must find the equation

$$\Box h_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^\alpha\partial_\mu h_{\alpha\nu} - \partial^\alpha\partial_\nu h_{\alpha\mu}$$

$$- \eta_{\mu\nu}\Box h + \partial_\mu\partial_\nu h = -16\pi G_N T_{\mu\nu}$$

So, somehow gravitation is embodied in $h_{\mu\nu}$, with h_{00} being related (in a fixed frame) to the Newtonian potential φ.

If this is correct then there is one elementary test of the idea. Given a (weak) Newtonian potential φ, is there a spacetime metric for which geodesic motion reduces approximately to the Newtonian force law

$$\ddot{x}^i = -\partial_i \varphi \qquad ?$$

For this, let us ~~ta~~ consider a metric

$$ds^2 = -(1+2\varphi)\,dt^2 + (1-2\varphi)\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right)$$

(This arises as follows. If $\bar{h}_{00} = -4\varphi$ and $\bar{h}_{ii} = 0$, then $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h_{00} = +4\varphi$.

Therefore $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$

$$\Rightarrow \quad h_{00} = -4\varphi + \frac{1}{2}(+4\varphi) = -2\varphi$$

$$h_{ii} = \bar{h}_{ii} - \frac{1}{2}\bar{h} = -2\varphi \quad)$$

Given this metric, ~~an~~ and neglecting any higher-order effects in φ, we now show that the geodesic equation of motion:

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu \dot{x}^\lambda = 0$$ reproduces the Newtonian law: $\ddot{x}^i = -\partial_i \varphi$.

Before giving the proof, we need to tackle the question of what is the "dot" in $\dot{x}^\mu$. In special relativity, particle trajectories are labelled by a parameter τ that is arbitrary. We can re-parametrise the equations as we like. (recall: $\int d\tau \sqrt{\dot{x}^\mu \dot{x}_\mu}$ is the action)

One choice of τ is to identify it with the actual time coordinate of the particle. In that case,

$\tau = t$ and $\frac{dx^0}{d\tau} = 1$. ~~and so that~~

Now $\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda = 0$

$$\longrightarrow \quad \ddot{x}^i + \Gamma^i_{00} + 2\Gamma^i_{0j}\dot{x}^j + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$$

Now for the metric we have chosen, Γ^i_{0j}. Also $\Gamma^i_{jk}\dot{x}^j\dot{x}^k$ will turn out to be higher-order in φ so we neglect it. Finally,

$$\Gamma^i_{00} = \frac{1}{2} g^{ii}\left(g_{i0,0} + g_{i0,0} - g_{00,i}\right)$$

$$= -\frac{1}{2}\left(1+2\Phi\right)\left(-2\Phi_{,i}\right)$$

$$= +\partial_i\Phi + O(\varphi^2)$$

Thus $\ddot{x}^i + \partial_i\varphi = 0$ as desired!

We can justify neglecting $\Gamma^i_{jk}\dot{x}^j\dot{x}^k$ as follows. In $\ddot{x}^i + \partial_i\varphi = 0$, $x \to \lambda x$ and $\varphi \to \lambda^2\varphi$ preserves the eqn.

~~Pause~~ the terms $\ddot{x}^i$ and Γ^i_{00} are both of order λ. What is the order of $\Gamma^i_{jk}\dot{x}^j\dot{x}^k$?

$$x \to \lambda x \quad , \quad \Gamma^i_{jk} \sim g_{ij,k} \sim \partial_i \Phi \sim \frac{\lambda^2}{\lambda}$$

So ~~t~~ this term is of $O(\lambda^3)$ and can be neglected.