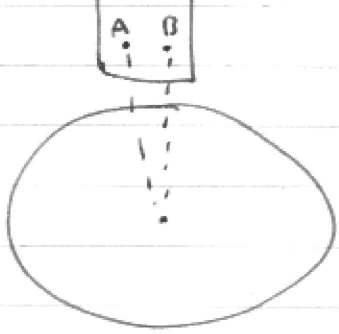
Lecture 4. 18/8/09

Inertial frames and the principle of equivalence.

A physical approach to General Relativity can be provided by some thought experiments on falling bodies. These follow from the equivalence of gravitational and inertial mass.
It turns out that this approach, too leads to (pseudo) Riemannian geometry. And it will provide a way to think about the physics of gravitation

Einstein started by noting that a freely falling observer experiences no gravitational field. Indeed such an observer, enclosed inside a freely falling sealed lift, would not even know there was a gravitational field.

There is a slight catch. Inhomogeneities in the gravitational field would show up:



A, B come closer together as the lift falls.

Therefore the principle of equivalence can hold only over very small regions of space and time.

The principle of equivalence ~~basically~~ amounts to the following statement. In a gravitational field (non-trivial spacetime metric) an observer satisfies

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$$

If the observer is seated inside a "freely falling lift", then since the lift also follows the above geodesic, the observer is weightless (experiences no gravitational force) inside the lift. Therefore, for a localised observer, the effect of gravity can be "cancelled" by going to a freely falling frame of reference.

Since this freely falling frame amounts to a new choice of coordinates x'^μ, what we are saying is that there always exist coordinates $x'^\mu(x)$ such that

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0 \iff \ddot{x}'^\mu = 0$$

Is this a true statement about Riemannian geometry? If so, it can be borrowed for the pseudo-Riemannian case relevant here.

Consider a 2-dimensional space (e.g. a 2-sphere) on which a particle follows

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \qquad (i,j,k = 1,2)$$

Intuitively we can guess the following. In the infinitesimal neighbourhood of a point x^μ_0 it should be possible to choose

$$g_{ij}(x^\mu_0) = \delta_{ij}$$

but this may not be extendable away from that point. It can be shown that in fact we can also make

$$g_{ij,k}(x^\mu_0) = 0$$

by a choice of coordinates, but we cannot make 2nd derivatives vanish (eg $g_{ij,kl}$).

Let's show this explicitly. Since

$$g'_{kl} = \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} g_{ij}$$

we can fix a point P and denote $\frac{\partial x^j}{\partial x'^l}(P) = M^j{}_l$

Then $g'_{kl}(P) = (M^T g M)_{kl}$

But since g is real symmetric, an orthogonal matrix M can always be chosen so that

$$(M^T g M)_{kl} = \text{diagonal.}$$

Later we can make the diagonal entries $= 1$

Next we ask, can we make $g_{ij,k} = 0$ at the point P? For this we show instead that the entire Christoffel symbol

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(g_{lj,k} + g_{lk,j} - g_{jk,l} \right)$$

can be made to vanish at a point.

Let us first ask what is the transformation law of Γ under a change of coordinates.

$$g'_{ij}(x') = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} g_{kl}$$

It is a tedious but straightforward exercise to show that under this transformation,

$$\Gamma'^i_{jk}(x') = \frac{\partial x'^i}{\partial x^l} \frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^k} \Gamma^l_{mn} + \frac{\partial^2 x^l}{\partial x'^j \partial x'^k} \frac{\partial x'^i}{\partial x^l}$$

Because of the extra inhomogeneous term, the Christoffel symbol does <u>not</u> transform like a tensor.

Now fix a point P ~~[illegible]~~ and assume we are already in coordinates x^i in which $g_{ij}(P) = \delta_{ij}$. ~~[illegible]~~ Also choose x^i such that the point P is $x^i = 0$.

Now define

$$x'^i = x^i - \frac{1}{2}\Gamma^i_{jk}(P)\, x^j x^k$$

~~Then~~ $\cancel{\Gamma'^i_{jk}(P) =}$

$$\left.\frac{\partial x'^i}{\partial x^l}\right|_P = \delta^i{}_l$$

$$\left.\frac{\partial^2 x'^i}{\partial x^l \partial x^m}\right|_P = -\Gamma^i_{jk}(P).$$

Thus $\Gamma'^i_{jk}(P) = \Gamma^i_{jk}(P) - \Gamma^i_{jk}(P) = 0.$

~~Th~~ Thus we have proved that that there is always <u>some</u> coordinate system for which the geodesic equation at a point P is

$$\ddot{x}^i = 0$$

Note also that, defining

$$\cancel{\Gamma_{ijk}} \quad \Gamma_{ijk} = g_{il}\Gamma^l_{jk}$$

$$= \frac{1}{2}\left(g_{ij,k} + g_{ik,j} - g_{jk,i}\right)$$

we have:

$$\Gamma_{ijk} + \Gamma_{jik} = \frac{1}{2}\left(g_{ij,k} + g_{ik,j} - g_{jk,i}\right)$$

$$+ \frac{1}{2}\left(g_{ji,k} + g_{jk,i} - g_{ik,j}\right)$$

$$= g_{ij,k}.$$

Therefore vanishing of $\Gamma^{i}_{jk}(P)$ also implies vanishing of $g_{ij,k}(P)$, i.e. all first derivatives of the metric.

One can convince oneself that 2nd derivatives cannot similarly be made to vanish.

$$g'_{ij} = \delta_{ij} \to \frac{d(d+1)}{2} \text{ conditions} \qquad \frac{\partial x^{i'}}{\partial x^j} : d^2 \text{ nos.}$$

$$g'_{ij,k} = 0 \to \frac{d^2(d+1)}{2} \quad " \qquad \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k} \quad \frac{d^2(d+1)}{2} \text{ nos.}$$

$$g'_{ij,kl} = 0 \to \frac{d^2(d+1)^2}{4} \qquad \frac{\partial^3 x^{i'}}{\partial x^j \partial x^k \partial x^l} \quad \frac{d^2(d+1)(d+2)}{6} \text{ nos.}$$

Now $\dfrac{d^2(d+1)(d+2)}{6} > \dfrac{d^2(d+1)^2}{4}$

and it gets worse -- so we see that 2nd derivatives of g_{ij} cannot be fixed to some chosen value by coordinate transformations.