Lecture 6 1/9/09

Having had some experience of spacetime, we can go back to curved space and try to understand what curvature means. For this, as expected we will have to consider second-order variations of the metric.

Before getting into that, let us revisit our old friend, the geodesic equation, and recast it in a few different ways.

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$$

$$\rightarrow \quad \frac{d}{dt}\frac{dx^i}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

$$\underbrace{\left(\frac{d}{dt}\delta^i{}_k + \Gamma^i_{jk}\frac{dx^j}{dt}\right)}\frac{dx^k}{dt} = 0.$$

a generalisation of $\frac{d}{dt}\delta^i{}_k$ by the addition of an extra, metric-dependent term.

Define $\left(\frac{D}{Dt}\right)^i{}_k = \frac{d}{dt}\delta^i{}_k + \Gamma^i_{jk}\frac{dx^j}{dt}$

Then the geodesic eqn becomes

$$\frac{D}{Dt}\frac{d}{dt}x^i = 0$$

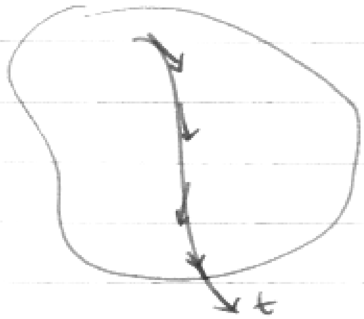
$\frac{D}{Dt}$ is called the "covariant derivative along the geodesic".

In flat space it just reduces to the ordinary derivative along the geodesic.

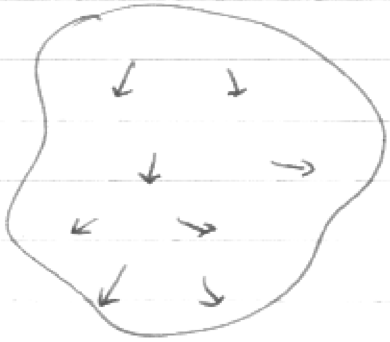
Also, $\frac{d^2}{dt^2}x^i = 0$ is not preserved by general coord. transfs. But $\frac{D}{Dt}\frac{dx}{dt} = 0$ is preserved.

Next, note the following. If $x^i(t)$ is the ~~tr~~ trajectory of a particle ~~then~~ then $\frac{dx^i}{dt} = V^i$ is the velocity vector. It is defined at all points ~~of the space~~ along the trajectory.

This is a special case of a vector <u>field</u>, defined at all points of a space.



velocity vector field



general vector field.

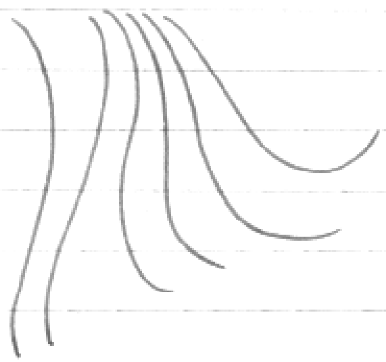
Suppose we take a general vector field w^i and consider its derivative $\frac{dw^i}{dt}$ along a fixed trajectory.

Clearly,
$$\frac{dw^i}{dt} = \frac{dx^j}{dt}\frac{\partial}{\partial x^j} w^i = v^j \partial_j w^i$$

where v^j is the velocity vector along the trajectory.

If w^i is itself the velocity vector v^i ~~then~~ then
$$\frac{dv^i}{dt} = v^j \partial_j v^i .$$

Indeed we may consider a <u>family</u> of geodesics:



etc.

~~As along~~ the v^i is defined all along this family so it becomes a full-fledged vector field. The v^i so defined, along any specific geodesic, satisfy:

$$v^j \partial_j v^i + \Gamma^i_{jk} v^j v^k = 0$$

or $$v^j \left(\partial_j v^i + \Gamma^i_{jk} v^k \right) = 0$$

The quantity in brackets is a covariant derivative along an arbitrary direction x^j:

$$\boxed{(D_j V)^i = \partial_j v^i + \Gamma^i_{jk} v^k}$$

Thus the geodesic eqn is $v^j D_j v^i = 0$. A geometrical interpretation of this equation is as follows. Given an arbitrary vector field w^i, the operation $v^j D_j$ on it produces a new vector field which is the covariant change of w^i along the trajectory whose velocity vector is v^i.

If $v^j D_j w^i = 0$, w^i is said to be parallel-transported along the trajectory. In this interpretation, the geodesic equation

$v^j D_j v^i = 0$ means that:

"a geodesic is a trajectory whose velocity vector is parallel-transported along it."

It is straightforward to check that if w^i is a vector, ie

$$w'^i(x') = \frac{\partial x'^i}{\partial x^j} w^j$$

then $$D'_j w'^i(x') = \frac{\partial x^l}{\partial x'^j} \frac{\partial x'^i}{\partial x^k} D_l w^k$$

Therefore our formulas are completely covariant under coordinate transformations.

Note also that given a number of vector fields $w_1^i, w_2^i \dots$ we can define the covariant derivative on their product by requiring distributivity:

$$D_k(w_1^i w_2^j) = (D_k w_1^i) w_2^j + w_1^i D_k w_2^j$$

$$= (\partial_k w_1^i + \Gamma^i_{kl} w_1^l) w_2^j$$

$$+ w_1^i (\partial_k w_2^j + \Gamma^j_{kl} w_2^l)$$

$$= \partial_k(w_1^i w_2^j) + \Gamma^i_{kl} w_1^l w_2^j + \Gamma^j_{kl} w_1^i w_2^l$$

Indeed, any quantity T^{ij} that transforms like the product of two vector fields, will have a covariant derivative:

$$D_k T^{ij} = \partial_k T^{ij} + \Gamma^i_{k\ell} T^{\ell j} + \Gamma^j_{k\ell} T^{i\ell}$$

This is easily extended to $T^{i_1 \cdots i_n}$.

Next, suppose we consider the gradient of a function f: $\dfrac{\partial f}{\partial x^i} = \partial_i f$.

Since f is a scalar, we easily see that this transforms as:

$$\partial'_i f' = \frac{\partial x^j}{\partial x'^i} \partial_j f$$

What should be the covariant derivative $D_j \partial_i f$? We determine it by requiring:

$$\partial_j (\partial_i f \, W^i) = D_j \partial_i f \, . \, W^i + \partial_i f \, . \, D_j W^i$$

We already know ~~the~~ $D_j W^i$, and we know that on the LHS, the derivative of a scalar fn is just the ordinary derivative.

We find: $\partial_j \partial_i f \; W^i + \cancel{\partial_i f \, \partial_j W^i}$

$$= (D_j \, \partial_i f) W^i + \cancel{\partial_i f \, \partial_j W^i} + \partial_i f \, \Gamma^i_{jk} W^k$$

So $(D_j \, \partial_i f) W^i = (\partial_j \partial_i f) W^i - \partial_i f \, \Gamma^i_{jk} W^k$

$$= (\partial_j \partial_i f - \Gamma^\ell_{ji} \partial_\ell f) W^i .$$

Since this is true for arbitrary w^i, we have:

$$D_j \partial_i f = \partial_j \partial_i f - \Gamma^k_{ji} \partial_k f$$

This in turn gives us a rule to covariantly differentiate any "covariant vector" a_i:

$$D_j a_i = \partial_j a_i - \Gamma^k_{ji} a_k$$

Again this can be extended to any tensor with arbitrarily many lower indices.

Now we can covariantly differentiate the metric itself! We get

$$D_k g_{ij} = \partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il}$$

Notice that the RHS is made up entirely of first derivatives of g_{ij}. But by a coordinate transformation we can make all first derivatives vanish at a point. Therefore at any point, $D_k g_{ij} = 0$ in <u>some</u> coordinate system.

However by construction, $D_k g_{ij}$ is a tensor: it transforms homogeneously:

$$D'_k g'_{ij}(x') = \frac{\partial x^n}{\partial x'^k} \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} D_n g_{lm}$$

Therefore it must be identically zero! Explicit calculation confirms that $\boxed{D_k g_{ij} = 0}$.

We can finally address the question of curvature. Basically, we need a tensor involving second derivatives of the metric.

Since $D_i g_{jk} = 0$, we cannot use $D_i D_j g_{kl}$!.

Instead, consider the following process. Take a vector field w^i and perform two covariant differentiations on it to get $D_i D_j w^k$. Of course the result depends on derivatives of w^k at a point P. However if we antisymmetrise in i and j, the term $\partial_i \partial_j w^k$ drops out and we get an answer proportional to w^k. Indeed we find:

$$D_i D_j w^k = D_i (\partial_j w^k + \Gamma^k_{jl} w^l)$$

$$= \partial_i \partial_j w^k + \Gamma^k_{jl,i} w^l + \Gamma^k_{jl} \partial_i w^l$$

$$- \Gamma^l_{ij} \partial_l w^k - \Gamma^m_{ij} \Gamma^k_{ml} w^l$$

$$+ \Gamma^k_{il} \partial_j w^l + \Gamma^k_{im} \Gamma^m_{jl} w^l$$

Thus $D_i D_j w^k - D_j D_i w^k$

$$= \left(\Gamma^k_{jl,i} - \Gamma^k_{il,j} + \Gamma^k_{im} \Gamma^m_{jl} - \Gamma^k_{jm} \Gamma^m_{il} \right) w^k$$

$$\equiv R^k{}_{lij} w^l$$

$$R^k{}_{lij} = \Gamma^k_{jl,i} - \Gamma^k_{il,j} + \Gamma^k_{im} \Gamma^m_{jl} - \Gamma^k_{jm} \Gamma^m_{il}$$

$R^i{}_{jkl}$ is called the Riemann curvature tensor.

Physical interpretation:

What does it mean to compute $D_i D_j w^k - D_j D_i w^k$

Suppose we parallel transport the vector w^i around a closed loop of infinitesimal size:

We would like to compute

$$\delta w^i = \tilde{w}^i - w^i \quad \text{(in the figure)}$$

If t parametrises the curve then

$$\delta w^i = \oint dt \left(\frac{dw^i}{dt} \right) = - \oint dt \, \Gamma^i_{jk} \frac{dx^j}{dt} w^k$$

where in the last ~~term~~ equality we have used the parallel transport equation.

Now Stokes' theorem tells us that

$$\oint f_i \, dx^i = \frac{1}{2} \int \left(\partial_j f_i - \partial_i f_j \right) d\Sigma^{ij}$$

where $d\Sigma^{ij}$ is the area element of the region bounded by the contour.

~~Thus~~ For us,

$$\oint f_i \, dx^i = -\oint (\Gamma^i_{jk} w^k) \, dx^j$$

So ~~~~ $\delta w^i = \frac{1}{2} \int \left[\partial_l (\Gamma^i_{mk} w^k) - \partial_m (\Gamma^i_{lk} w^k) \right] d\Sigma^{lm}$

$$\underset{\text{(for small loop)}}{\simeq} \frac{1}{2} \Big[\Gamma^i_{mk,l} w^k - \Gamma^i_{lk,m} w^k + \Gamma^i_{mk} \partial_l w^k - \Gamma^i_{lk} \partial_m w^k \Big] \Delta\Sigma^{lm}$$

$$= \frac{1}{2} \Big[(\Gamma^i_{mk,l} - \Gamma^i_{lk,m}) w^k + (-\Gamma^i_{mk} \Gamma^k_{ln} w^n + \Gamma^i_{lk} \Gamma^k_{mn} w^n) \Big] \Delta\Sigma^{lm}$$

$$= \frac{1}{2} \Big[\Gamma^i_{mk,l} - \Gamma^i_{lk,m} + \Gamma^i_{lj} \Gamma^j_{mk} - \Gamma^i_{mj} \Gamma^j_{lk} \Big] w^k \Delta\Sigma^{lm}$$

So $\delta w^i = \frac{1}{2} R^i{}_{klm} w^k \Delta\Sigma^{lm}$

We see that $R^i{}_{jkl}$ defines the linear transformation undergone by the vector after parallel transport around a small loop.