Lecture 7 3/9/09

We have seen that:

$$R^i{}_{klm} = \Gamma^i_{mk,l} - \Gamma^i_{lk,m} + \Gamma^i_{lj}\Gamma^j_{mk} - \Gamma^i_{mj}\Gamma^j_{lk}$$

It is convenient to lower one index:

$$R_{jklm} = g_{ij} R^i{}_{klm}$$

$$= \tfrac{1}{2}\big(g_{jm,kl} + g_{kl,jm} - g_{jl,km} - g_{km,jl}$$

$$+ \text{ terms involving } (g_{ij,k})^2 \big)$$

Let us choose coordinates so that $g_{ij,k} = 0$ at a point. We have seen earlier that such coordinates are possible. They are called Riemann normal coordinates (RNC) at the point.

It is sufficient to work in this coord basis to study the symmetries of R_{jklm}. From the above, we see that

$$R_{jklm} = -R_{jkml}$$

$$R_{jklm} = -R_{kjlm}$$

$$R_{jklm} = R_{lmjk}$$

$$R_{jklm} + R_{jlmk} + R_{jmkl} = 0.$$

We also note that from R_{jklm} we can extract:

$$R_{km} = g^{jl} R_{jklm} = R^{i}{}_{kim}$$

This satisfies $R_{km} = R_{mk}$ and is called the Ricci tensor.

Finally $R = g^{km} R_{km} = g^{jl} g^{km} R_{jklm}$

is a scalar called the Ricci scalar.

It can easily be shown that

$$D_i R_{jklm} + D_j R_{kilm} + D_k R_{ijlm} = 0$$

(Bianchi identity).

For this, ~~take~~ evaluate the first term above in Riemann normal coordinates. Then

$$D_i R_{jklm} = \partial_i R_{jklm} \qquad (\text{since } \Gamma^{i}_{jk} = 0 \text{ at } P)$$

$$= \frac{1}{2} \left(g_{jm,kli} + g_{kl,jmi} - g_{jl,kmi} - g_{km,jli} \right.$$

(Note that differentiating terms like $g_{ij,k}\, g_{lm,n}$ leads to $g_{ij,kp}\, g_{lm,n}$ etc. which is still zero in RNC).

Thus in RNC we have

$$D_i R_{jklm} + D_j R_{kilm} + D_k R_{ijlm}$$

$$= \frac{1}{2}\Big(\cancel{g_{jm,kli}} + \cancel{g_{kl,jmi}} - \cancel{g_{jl,kmi}} - \cancel{g_{km,jli}}$$

$$+ \cancel{g_{km,ilj}} + \cancel{g_{il,kmj}} - \cancel{g_{kl,imj}} - \cancel{g_{im,klj}}$$

$$+ \cancel{g_{im,jlk}} + \cancel{g_{jl,imk}} - \cancel{g_{il,jmk}} - \cancel{g_{jm,ilk}} \Big)$$

$$= 0 .$$

By contracting the above identity with $g^{il} g^{km}$ (and keeping in mind that $D_i g_{jk} = 0$ therefore also $D_i g^{jk} = 0$) we find:

$$D_i R - 2 D^j R_{ij} = 0$$

or equivalently:

$$D^j \left(R_{ij} - \frac{1}{2} g_{ij} R\right) = 0$$

This is sometimes also called the Bianchi identity. Defining the "Einstein tensor"

$$G_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij} R$$

we see that $D^i G_{ij} = 0$.

Uniqueness

Upto combinations, R_{ijkl}, R_{ij} and R are the only tensors that can be made out of second derivatives of g_{ij}.

(of course we can consider $R_{ijkl} + g_{ik} R_{jl}$ etc. which is why we said "upto combinations".)

Now let us go back to our original problem. We wanted ~~to~~ generalise Newtonian gravity to make it consistent with special relativity for weak gravitational fields. We ended up with the approximate equation: (see Page (17)):

$$\Box h_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta} - \partial^\alpha \partial_\mu h_{\nu\alpha} - \partial^\alpha \partial_\nu h_{\mu\alpha}$$

$$- \eta_{\mu\nu} \Box h + \partial_\mu \partial_\nu h = -16\pi G T_{\mu\nu} \qquad (*)$$

where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $O(h_{\mu\nu})^2$ is neglected (linearised approximation).

Now we realise that, being a second-order derivative of $g_{\mu\nu}$, the LHS has to be built out of $R_{\mu\nu\lambda\rho}$, $R_{\mu\nu}$ and R. The unique 2nd-rank symmetric tensors linear in these are $R_{\mu\nu}$ and $g_{\mu\nu} R$.

Finally since $T_{\mu\nu}$ is conserved in flat space: $\partial^\mu T_{\mu\nu} = 0$, we ~~[illegible]~~ expect it to be conserved also in curved space:

$$D^\mu T_{\mu\nu} = 0.$$

The unique combination of $R_{\mu\nu}$ and $g_{\mu\nu}R$ that is conserved is $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.

Thus upto a constant, the LHS of Eqn ⊛ on the previous page must be proportional to $G_{\mu\nu}$! The constant is easily found to be $-\frac{1}{2}$, ie:

$$R_{\mu\nu} \sim -\frac{1}{2} \Box h_{\mu\nu} + \ldots$$

In fact $R_{\mu\nu} = -\frac{1}{2} \Box h_{\mu\nu} + \frac{1}{2} (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$

where $\xi_\mu = \partial^\alpha h_{\alpha\mu} - \frac{1}{2} \partial_\mu h^\alpha{}_\alpha$

Thus $R_{\mu\nu} = -\frac{1}{2} \Big(\Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial^\alpha h_{\alpha\nu}$

$- \partial_\nu \partial^\alpha h_{\alpha\mu} \Big)$

It follows that $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is precisely equal to $-\frac{1}{2}$ of our LHS of eq ⊛.

Hence we propose the Einstein equation

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \, T_{\mu\nu}}$$

which is the fundamental eqn of General Relativity

Examples of curvature tensor

i) 2-sphere: $ds^2 = r^2(d\theta^2 + \cos^2\theta\, d\varphi^2)$

We saw that in these coordinates,

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\varphi_{\theta\theta} = 0$$

$$\Gamma^\theta_{\varphi\varphi} = \frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$$

$$\Gamma^\varphi_{\theta\varphi} = -\tan\theta$$

Now the **only nontrivial** component of R_{ijkl} is $R_{\theta\varphi\theta\varphi} = g_{\theta\theta} R^\theta{}_{\varphi\theta\varphi}$

$$= r^2\left(\Gamma^\theta_{\varphi\varphi,\theta} - \Gamma^\theta_{\theta\varphi,\varphi} + \Gamma^\theta_{\theta\varphi}\Gamma^\varphi_{\varphi\varphi} - \Gamma^\theta_{\varphi\varphi}\Gamma^\varphi_{\theta\varphi}\right)$$

$$= r^2\left(\cos^2\theta - \sin^2\theta - 0 + 0 - \sin\theta\cos\theta \cdot \frac{-\sin\theta}{\cos\theta}\right)$$

$$= r^2\cos^2\theta$$

Thus $R_{\theta\varphi\theta\varphi} = r^2\cos^2\varphi$

Note that $R_{\theta\varphi\theta\varphi}$ appears to vanish at $\theta = \pm\frac{\pi}{2}$, but these are bad points of the coordinate system! At these points, the φ coordinate is not well-defined.

Next $R_{\theta\theta} = g^{\varphi\varphi} R_{\theta\varphi\theta\varphi} = 1$, $R_{\varphi\varphi} = g^{\theta\theta} R_{\theta\varphi\theta\varphi}$
$= \cos^2\theta$

Finally $R = g^{\theta\theta} R_{\theta\theta} + g^{\varphi\varphi} R_{\varphi\varphi}$
$= \frac{1}{r^2}(1 + 1) = \frac{2}{r^2}$.

(ii) AdS_2

$$ds^2 = r^2(-\cosh^2\rho \, dt^2 + d\rho^2)$$

As we saw earlier, $\Gamma^{\rho}_{tt} = \sinh\rho \cosh\rho$

$$\Gamma^{t}_{\rho t} = \tanh\rho$$

So $R_{\rho t \rho t} = r^2(\Gamma^{\rho}_{tt,\rho} - \Gamma^{\rho}_{\rho t,t} + \Gamma^{\rho}_{\rho t}\Gamma^{t}_{tt} - \Gamma^{\rho}_{tt}\Gamma^{t}_{\rho t})$

$$= r^2(\cosh^2\rho + \sinh^2\rho - \sinh^2\rho)$$

$$= r^2\cosh^2\rho$$

Note that this is non-vanishing for all ρ.

Next $R_{\rho\rho} = g^{tt} R_{\rho t \rho t} = -1$

$$R_{tt} = g^{\rho\rho} R_{\rho t \rho t} = \cosh^2\rho$$

$$R = g^{\rho\rho}R_{\rho\rho} + g^{tt}R_{tt} = \frac{1}{r^2}(-1-1) = -\frac{2}{r^2}.$$