<u>Lecture 8</u> ~~15~~8/9/09

We have motivated Einstein's equations but have not said much about the RHS, the energy-momentum tensor. ~~Let us see a couple of examples~~. Let us now ask how to obtain Einstein's equations as Euler–Lagrange equations for an action. This will clarify the origin of $T_{\mu\nu}$ and much else.

Imagine a Lagrangian $\mathcal{L}(g, \partial g \ldots)$ whose eqn of motion is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 .$$

Since LHS is 2nd-derivative in $g_{\mu\nu}$, $\mathcal{L}$ must also be 2nd-derivative. Also $\mathcal{L}$ must be a <u>scalar</u> <u>density</u> so that $\int d^4x\, \mathcal{L}$ can be general coordinate invariant.

We know that $\sqrt{g}\, d^4x$ is coord-invariant, so

$$S = \int d^4x \sqrt{g}\, X = \int d^4x\, \mathcal{L}$$

where X is a scalar. Thus $\mathcal{L} = \sqrt{g}\, X$.

The unique scalar made out of 2nd derivatives of $g_{\mu\nu}$ is the Ricci scalar R, therefore

$$S \sim \int d^4x \sqrt{g}\, R .$$

Before fixing the constant, let us check that this action gives the correct equations of motion.

We need to vary $\sqrt{g}\,R = \sqrt{g}\,g^{\mu\nu}R_{\mu\nu}$.
Then

$$\delta(\sqrt{g}\,R) = \delta(\sqrt{g})\,R + \sqrt{g}\,\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{g}\,g^{\mu\nu}\,\delta R_{\mu\nu}$$

Now we use the identity:

$$-\tfrac{1}{2}g_{\mu\nu}\,\delta g^{\mu\nu} = \sqrt{g}^{\,-1}\,\delta\sqrt{g}$$

Proof: $\sqrt{g} = e^{\frac{1}{2}\operatorname{tr}\ln \underset{\sim}{g}} = e^{-\frac{1}{2}\operatorname{tr}\ln \underset{\sim}{g}^{-1}}$

$$\delta\sqrt{g} = -\tfrac{1}{2}\operatorname{tr}\,\underset{\sim}{g}\,\delta\underset{\sim}{g}^{-1}\cdot\sqrt{g}$$

$$= -\tfrac{1}{2}\,g_{\mu\nu}\,\delta g^{\mu\nu}\cdot\sqrt{g}$$

Hence $-\tfrac{1}{2}g_{\mu\nu}\,\delta g^{\mu\nu} = \sqrt{g}^{\,-1}\,\delta\sqrt{g}$

Therefore $\delta\sqrt{g} = \sqrt{g}\left(-\tfrac{1}{2}g_{\mu\nu}\,\delta g^{\mu\nu}\right)$

Thus
$$\delta(\sqrt{g}\,R) = \sqrt{g}\left(-\tfrac{1}{2}g_{\mu\nu}R\,\delta g^{\mu\nu}\right) + \sqrt{g}\,\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{g}\,g^{\mu\nu}\,\delta R_{\mu\nu}$$

$$= \sqrt{g}\left(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \sqrt{g}\,g^{\mu\nu}\,\delta R_{\mu\nu}$$

Now $\delta R_{\mu\nu}$ is quite complicated if we just compute it directly. The following observation makes things much easier:

Although $\Gamma^\alpha_{\mu\nu}$ is not a tensor, the infinitesimal variation $\delta\Gamma^\alpha_{\mu\nu}$ <u>is</u> a tensor (the inhomogeneous terms in the transformation law, being metric-independent, cancel out). Therefore it makes sense to define

$$D_\alpha \Gamma^\beta_{\gamma\delta} \equiv \Gamma^\beta_{\gamma\delta,\alpha} - \Gamma^\rho_{\alpha\gamma}\Gamma^\beta_{\rho\delta} - \Gamma^\rho_{\alpha\delta}\Gamma^\beta_{\gamma\rho} + \Gamma^\beta_{\alpha\rho}\Gamma^\rho_{\gamma\delta}$$

Now $R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\nu\beta}\Gamma^\beta_{\mu\alpha}$

$$\delta R_{\mu\nu} = \delta\Gamma^\alpha_{\mu\nu,\alpha} - \delta\Gamma^\alpha_{\mu\alpha,\nu} + \delta\Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} + \Gamma^\alpha_{\alpha\beta}\delta\Gamma^\beta_{\mu\nu} - \delta\Gamma^\alpha_{\nu\beta}\Gamma^\beta_{\mu\alpha} - \Gamma^\alpha_{\nu\beta}\delta\Gamma^\beta_{\mu\alpha}$$

$$= D_\alpha \delta\Gamma^\alpha_{\mu\nu} - D_\nu \delta\Gamma^\alpha_{\mu\alpha}$$

(this can be checked explicitly, but also follows easily in RNC since $\delta R_{\mu\nu}$ and $\delta\Gamma^\alpha_{\mu\nu}$ are both **tensors** and

$$\delta R_{\mu\nu} = \delta\Gamma^\alpha_{\mu\nu,\alpha} - \delta\Gamma^\alpha_{\mu\alpha,\nu} \quad \text{in RNC.})$$

Then $\sqrt{g}\, g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{g}\, g^{\mu\nu}\left(D_\alpha\delta\Gamma^\alpha_{\mu\nu} - D_\nu\delta\Gamma^\alpha_{\mu\alpha}\right)$

$$= \cancel{\sqrt{g}\, D_\alpha\left(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\nu}\right.}$$

Now $\sqrt{g}\, g^{\mu\nu} D_\alpha \delta\Gamma^\alpha_{\mu\nu}$

$$= \sqrt{g}\, \underbrace{D_\alpha \left(g^{\mu\nu} \delta\Gamma^\alpha_{\mu\nu} \right)}$$

$$= \partial_\alpha (\quad) + \Gamma^\beta_{\alpha\beta}\, g^{\mu\nu} \delta\Gamma^\beta_{\mu\nu}$$

$$= \partial_\alpha \left(\sqrt{g}\, g^{\mu\nu} \delta\Gamma^\alpha_{\mu\nu} \right)$$

since $D_\alpha \sqrt{g} = 0 \Rightarrow \partial_\alpha \sqrt{g} = \Gamma_\alpha{}^\beta{}_\beta \sqrt{g}$

Similarly $-\sqrt{g}\, g^{\mu\nu} D_\nu \delta\Gamma^\alpha_{\mu\alpha}$

$$= -\partial_\nu \left(\sqrt{g}\, g^{\mu\nu} \delta\Gamma^\alpha_{\mu\alpha} \right)$$

So $\sqrt{g}\, g^{\mu\nu} \delta R_{\mu\nu}$ is a <u>total</u> <u>derivative</u> and does not contribute to the eqns of motion. (There are some subtleties which we may return to later).

$\therefore$ we get $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$ by varying $\int \sqrt{g}\, R\, d^4x$.

Now about the constant: we will choose it so that

$$S = -\frac{1}{16\pi G} \int \sqrt{-g}\, R\, d^4x$$

Next we need to introduce matter and define the energy-momentum tensor.
Suppose some field (Klein-Gordon field φ, or an electromagnetic field A_μ) is coupled to gravity. Then we must modify the usual Klein-Gordon action:

$$S = \int d^4x \left(-\frac{1}{2} \partial_\mu \varphi\, \partial^\mu \varphi - V(\varphi) \right)$$

and Maxwell action:

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

to be general-coordinate invariant when φ transforms as a scalar and A_μ as a vector. The minimal modification required is:

$$S_{KG} \rightarrow \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right)$$

$$S_{Maxwell} \rightarrow \int d^4x \sqrt{-g} \left(-\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \right)$$

The story is more complicated for fermions and we will address it later.

Given the above covariant matter action, we <u>define</u> the stress-energy tensor $T_{\mu\nu}$ by:

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{2} T_{\mu\nu}$$

As examples:

$$\cancel{T^{(KG)}_{\mu\nu} = \frac{2}{\sqrt{-g}} \left(\sqrt{-g} \cdot -\frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \cdot -\frac{1}{2} \sqrt{-g}\, g_{\mu\nu} \right.}$$

$$T^{(KG)}_{\mu\nu} = \frac{2}{\sqrt{-g}} \left\{ \sqrt{-g} \cdot -\frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \cdot -\frac{1}{2} \cdot g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \cdot \sqrt{-g}\, g_{\mu\nu} \right\}$$

$$= - \left(\partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \right)$$

Similarly,

$$T_{\mu\nu}^{(\mathrm{Maxwell})} = \frac{2}{\sqrt{-g}}\left(-\frac{1}{2}F_{\mu\alpha}F_{\nu\beta}\,g^{\alpha\beta}\sqrt{-g} - \frac{1}{4}\cdot -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu} \times g^{\alpha\beta}g^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta}\right)$$

$$= -\left(g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta} \right)$$

Now taking the full action:

$$S = S_G + \underbrace{S_{\mathrm{matter}}}_{\text{KG or Maxwell or any other}}$$

we have $\dfrac{\delta S}{\delta g^{\mu\nu}} = 0 \Rightarrow$

$$-\frac{1}{16\pi G}\left(\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\right) + \frac{1}{2}\sqrt{-g}\,T_{\mu\nu} = 0$$

$$\Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}}$$

as desired.

We can also now understand how conservation of $T_{\mu\nu}$ arises. If φ is a scalar field then

$$\frac{\delta S^{KG}}{\delta\varphi(x)} = 0 = \int d^4x \left(\frac{\delta\mathcal{L}}{\delta\varphi} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right)$$

are its equations of motion. Now if S^{KG} is general coordinate invariant then it depends also on g:

$$S^{KG} = S^{KG}(g_{\mu\nu}, \varphi)$$

Suppose now we make a general coord. transf.

Under this, $\delta g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu$ and

$$\delta g^{\mu\nu} = -\left(D^\mu \xi^\nu + D^\nu \xi^\mu\right).$$

The other fields in S_{matter}: φ, A_μ or whatever, all undergo some definite change of functional form (as a ~~vector~~ scalar, vector etc).

Then $$0 = \delta S = \int \frac{\delta S}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + \int \frac{\delta S}{\delta \varphi} \delta \varphi + \ldots$$

If the matter eqns of motion are satisfied then $\dfrac{\delta S}{\delta \varphi} = 0$ etc.

Hence $$\int \frac{\delta S}{\delta g^{\mu\nu}} \left(D^\mu \xi^\nu + D^\nu \xi^\mu\right) = 0$$

ie $$\int T_{\mu\nu} \left(D^\mu \xi^\nu + D^\nu \xi^\mu\right) \sqrt{g}\, d^4x = 0$$

$$\Rightarrow \int \left(D^\mu T_{\mu\nu}\right) \xi^\nu \sqrt{g}\, d^4x = 0$$

For this to be ~~true~~ with arbitrary $\xi^\mu(x)$, we must have

$$D^\mu T_{\mu\nu} = 0 \quad \text{pointwise in spacetime.}$$