lecture 19, 29/10/09

We can find a first integral as follows.

$$① \rightarrow \quad \ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a$$

$$② \rightarrow \quad \dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k$$

$$\rightarrow \quad 2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2a\rho\dot{a})$$

Inserting ①:

$$2\dot{a}\left(-\frac{4\pi G}{3}(\rho + 3p)a\right) = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2\rho a\dot{a})$$

Cancelling $\frac{8\pi G}{3}a$ from both sides $\Rightarrow$

$$\dot{a}(\rho + 3p) + 2\rho\dot{a} + \dot{\rho}a = 0$$

$$\Rightarrow \boxed{\frac{\dot{\rho}}{3(\rho + p)} = -\frac{\dot{a}}{a}}$$

It is easy to check that this is also the equation for conservation of $T_{\mu\nu}$: $D^\mu T_{\mu\nu} = 0$.

If we now have an equation of state, $f(\rho, p) = 0$, then we can determine ρ (and p) in terms of a. This can finally be plugged in to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

to determine $a(t)$.

A typical (and simple) eqn of state is

$$p = \underset{\substack{\downarrow \\ \text{const.}}}{w}\,\rho$$

Now $$\frac{\dot a}{a} = -\frac{\dot\rho}{3(\rho+p)}$$

$$= -\frac{\dot\rho}{3(1+w)\rho}$$

$$\Rightarrow \boxed{\rho = \text{const.}\; a^{-3(1+w)}}$$

Special examples: (i) Matter ("pressureless dust")
$\rightarrow p = 0$, ie $w = 0$

So $\rho \sim a^{-3}$

(roughly, decrease in number density of particles as the universe expands.)

(ii) Radiation: $T^\mu{}_\mu = F^{\mu\lambda}F_{\mu\lambda} - \frac{1}{4}\cdot 4\, F^{\alpha\beta}F_{\alpha\beta} = 0$

$$\Rightarrow p = \frac{\rho}{3}$$

Thus $w = \frac{1}{3}$ and $\rho \sim a^{-4}$

(roughly, decrease in density plus redshift)

(iii) Cosmological constant: $p = -\rho$ so $w = -1$

$$\Rightarrow \rho = \text{const} \quad \text{indep of } a(t).$$

Let us write the 2nd Friedmann eqn again:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

Define the Hubble parameter $H(t) = \frac{\dot{a}}{a}$

$(H(t = \text{now}) = H_0$ is the Hubble constant H_0.

The current value is 70 ± 10 km/sec/Mpc
↓
Mega parsec
$\simeq 3 \times 10^{24}$ cm

Now we have (dividing by $H(t)^2$)

$$1 = \frac{8\pi G}{3H(t)^2}\rho(t) - \frac{k}{\dot{a}^2}$$

We saw that ~~k<0~~ $k = \pm 1, 0$ (more generally any real value) are allowed. The sign of k in the above is determined by

$$\Omega \equiv \frac{8\pi G}{3H(t)^2}\rho(t) = \frac{\rho(t)}{\rho_{crit}(t)}$$

where $\rho_{crit} = \frac{3H^2}{8\pi G}$

Then $\Omega - \frac{k}{\dot{a}^2} = 1$

So $\Omega > 1 \Rightarrow k > 0$ (and $\rho > \rho_{crit}$)
$\Omega = 1 \Rightarrow k = 0$ (and $\rho = \rho_{crit}$)
$\Omega < 1 \Rightarrow k < 0$ (and $\rho < \rho_{crit}$)

Being dimensionless, $\Omega = \dfrac{\rho}{\rho_{crit}}$ is a very useful quantity.

We cannot proceed much further without the equation of state. So first let's assume all matter in the universe has $p = w\rho$ for a fixed w (later we will introduce a number of ~~[illegible]~~ species labelled by (ρ_i, p_i) with $p_i = w_i \rho_i$).

Then as we saw, $\rho = \text{const.}\ a^{-3(1+w)}$

Then $$\frac{8\pi G}{3\dot{a}^2} a^2 . a^{-3(1+w)} - \frac{k}{\dot{a}^2} = 1$$

$$\Rightarrow c\,\frac{8\pi G}{3} a^{-1-3w} - k = \dot{a}^2$$

Assume also that $k = 0$, then the equation is easily integrated to get

$$a(t) \sim t^{\frac{2}{3(1+w)}}$$

So the scale factor grows exponentially in t as long as $w > -1$.

We saw earlier that $w = -1$ is a possibility, corresponding to the cosmological term. In this case:

[illegible]

$$c\,\frac{8\pi G}{3} a^2 = \dot{a}^2 \Rightarrow \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G c}{3}}, \text{ const}$$
$$= H_0$$

Then $a(t) = e^{+H_0 t}$. (and $a \to 0$ as $t \to -\infty$)

Going back to $w > -1$, we have

$w = 0$ for matter ("pressureless dust") so

$\Rightarrow a(t) \sim t^{2/3}$ ("Einstein-deSitter" if $k=0$)

and $w = \frac{1}{3}$ for radiation $\Rightarrow a(t) \sim t^{1/2}$.
(again $a \to 0$ as $t \to 0$).

So far we have discussed all the possibilities for w but with $k = 0$. However this too can be easily incorporated by defining

$$\Omega_{curv} = \frac{-k}{\dot{a}^2}$$

and treating this (formally!) as a contribution to Ω. Then

$$\Omega + \Omega_{curv} = 1$$

where Ω comes from matter/radiation/cosm const.

Now suppose $\Omega = 0$ ("pure curvature") then

$$\dot{a}^2 = -k \quad \text{so} \quad k<0 \quad \text{and}$$

$$a = \sqrt{-k}\, t \quad \text{so} \quad a \sim t.$$

In fact this is of the form $a \sim t^{\frac{2}{3(1+w)}}$ for $w = -\frac{1}{3}$.

Thus we can simply think of an $\Omega = \sum_i \Omega_i$

$$= \sum_{i\,(phys)} \Omega_{i\,(phys)} + \Omega_{curv} = 1$$

where $p_i = w_i \rho_i$ and

i = matter $\rightarrow$ $w = 0$
= radiation $\rightarrow$ $w = \frac{1}{3}$
= cosm. const $\rightarrow$ $w = -1$
= curvature $\rightarrow$ $w = -\frac{1}{3}$.

Each of these separately is a toy "model" worth investigating. For example the curvature-dominated case $w = -\frac{1}{3}$ gives:

$$ds^2 = -dt^2 + t^2 \left(d\chi^2 + \sinh^2\chi \, (d\Omega_2^2) \right)$$

This spacetime is called the "Milne universe". It appears to pose a puzzle, since it looks like a nontrivial metric although $T_{\mu\nu} = 0$. Of course it's time-dependent, so maybe there's no puzzle.

In any case one can easily check that $R_{\mu\nu\lambda\rho} = 0$ for the Milne universe, so it's actually just flat Minkowski spacetime! (or maybe a part of it).

Another toy model that we just discussed is the purely cosmological one: $w = -1$ and

$$ds^2 = -dt^2 + e^{2H_0 t} (dx^2 + dy^2 + dz^2).$$

~~Referring back to our computation of Riemann ten Ricci tensors, we find~~

Notice that in the cosmological constant case, since $w = -1$ we have

$$T^\mu{}_\nu = \begin{bmatrix} -\rho & & & \\ & -\rho & & \\ & & -\rho & \\ & & & -\rho \end{bmatrix}$$

which suggests that our solution could be maximally symmetric as a 4d spacetime. Indeed, computation gives

$$R_{\mu\nu\lambda\rho} = k'(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda})$$

with $k' = H_0^2$

Thus we have a maximally symmetric spacetime with $k > 0$ in the 4d sense! This has to be de Sitter spacetime. We will discuss it later on.

Finally as our last "toy" example we consider trying to find a static universe using a combination of matter, radiation, cosmological constant and curvature.

We put $a(t) = \text{const}$ in the Friedmann eqns to find:

$$\frac{8\pi G}{3}\rho - \frac{k}{a^2} = 0$$

$$\frac{4\pi G}{3}(\rho + 3p) = 0$$

Hence $\frac{8\pi G}{3}\rho = \frac{k}{a^2}$ (a is const)

and $\rho + 3p = 0$. To achieve this we need a combination of pressureless matter & cosmological constant:

$$\rho_M + 3p_M + \rho_\Lambda + 3p_\Lambda = 0$$

$$= \rho_M - 2\rho_\Lambda = 0$$

$$\Rightarrow \rho_M = 2\rho_\Lambda .$$

The soln is called the Einstein static universe. It has $k > 0$.