10/2/2011 QM2

Addition of angular momenta.

Explicit example: two spin-$\frac{1}{2}$ particles.

~~A state can be~~ A basis for the possible states is $|m_1, m_2\rangle$ (we need not write s_1, s_2 since each one is fixed to be $\frac{1}{2}$)

Then $$S_z^{(1)}|m_1, m_2\rangle = \hbar m_1 |m_1, m_2\rangle$$

$$S_z^{(2)}|m_1, m_2\rangle = \hbar m_2 |m_1, m_2\rangle$$

(along with $\vec{S}^{(1)2}|m_1, m_2\rangle = \frac{3}{4}\hbar^2|m_1, m_2\rangle = \vec{S}^{(2)2}|m_1, m_2\rangle$)

There are four states altogether:

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle, \quad |\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \quad |-\tfrac{1}{2}, \tfrac{1}{2}\rangle, \quad |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

Now consider $\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$.

Clearly $[S_x, S_y] = i\hbar S_z$. Now $|m_1, m_2\rangle$ is an eigenstate of $\vec{S}^{(1)2}$, $\vec{S}^{(2)2}$, $S_z^{(1)}$, $S_z^{(2)}$. Since S_z commutes with all the above, these will also be eigenstates of S_z:

$$S_z|m_1, m_2\rangle = \hbar(m_1+m_2)|m_1, m_2\rangle.$$

However $\vec{S}^2$ does <u>not</u> commute with $S_z^{(1)}$, $S_z^{(2)}$:

$$\left[\vec{S}^2, S_z^{(1)}\right] = \left[\vec{S}^{(1)2} + \vec{S}^{(2)2} + 2S^{(1)}\cdot S^{(2)}, S_z^{(1)}\right]$$ (2)

$$= 2S_i^{(2)}\left[S_i^{(1)}, S_z^{(1)}\right]$$

$$= 2i\hbar\, S_i^{(2)} \epsilon_{i3k} S_k^{(1)}$$

$$= 2i\hbar\left(S_y^{(2)}S_x^{(1)} - S_x^{(2)}S_y^{(1)}\right)$$

But it <u>does</u> commute with $\vec{S}^{(1)2}$, $\vec{S}^{(2)2}$;

$$\left[\vec{S}^2, \vec{S}^{(1)2}\right] = \left(\vec{S}^{(1)2} + \vec{S}^{(2)2} + 2\vec{S}^{(1)}\cdot\vec{S}^{(2)}, \vec{S}^{(1)2}\right]$$

$$= 2S_i^{(2)}\left[S_i^{(1)}, \vec{S}^{(1)2}\right]$$

$$\overset{\shortparallel}{0}.$$

Hence we can transform ~~to~~ to a basis of eigenfunctions of $\vec{S}^{(1)2}$, $\vec{S}^{(2)2}$, $\vec{S}^2$, S_z (from eigenfns of $\vec{S}^{(1)2}$, $S_z^{(1)}$, $\vec{S}^{(2)2}$, $S_z^{(2)}$).

The new basis is denoted $|S, M\rangle$ with

$$\vec{S}^2|S,M\rangle = \hbar^2 S(S+1)|S,M\rangle$$

$$S_z|S,M\rangle = \hbar M|S,M\rangle$$

as well as $\vec{S}^{(1)2}|S,M\rangle = \vec{S}^{(2)2}|S,M\rangle = \frac{3}{4}\hbar^2|S,M\rangle$

Now, recall that $|m_1, m_2\rangle$ were already eigenstates of S_z with eigenvalues $\hbar(m_1+m_2)$. This equals $\hbar, 0, 0, -\hbar$

$(\frac{1}{2},\frac{1}{2})$ $(\frac{1}{2},-\frac{1}{2})$ $(-\frac{1}{2},\frac{1}{2})$ $(-\frac{1}{2},-\frac{1}{2})$

Hence the allowed values of M are $\pm 1, 0$, with 0 being doubly degenerate.

It follows that the four states $|S, M\rangle$ must correspond to

$$|S=1, M=1,0,-1\rangle \quad \text{and} \quad |S=0, M=0\rangle.$$

Suppose we want to write $\vec{S}^{\,2}$ as a 4×4-matrix in the $|m_1, m_2\rangle$ basis. For this we need to compute

$$\langle m_1, m_2 | \vec{S}^{\,2} | m_1, m_2\rangle$$

$$= \langle m_1, m_2 | \vec{S}^{(1)2} | m_1, m_2\rangle$$

$$+ \langle m_1, m_2 | \vec{S}^{(2)2} | m_1, m_2\rangle$$

$$+ 2\langle m_1, m_2 | \vec{S}^{(1)} \cdot \vec{S}^{(2)} | m_1, m_2\rangle$$

$$= \frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + 2\langle m_1, m_2 | S_z^{(1)} S_z^{(2)} | m_1, m_2\rangle$$

$$= \left(\frac{3}{4} + \frac{3}{4} \pm \frac{1}{2}\right)\hbar^2 = 2\hbar^2 \quad \text{for } |\tfrac{1}{2}, \tfrac{1}{2}\rangle \text{ and } |{-\tfrac{1}{2}}, -\tfrac{1}{2}\rangle$$

$$= \hbar^2 \quad \text{for } |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \text{ and } |{-\tfrac{1}{2}}, \tfrac{1}{2}\rangle$$

Thus
$$\langle\ | \vec{S}^{\,2} |\ \rangle = \hbar^2 \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The eigenvalues are $(2, 2, 2, 0)\hbar^2$.

(the submatrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has eigenvalues 2,0) ④

Thus we see that the structure is what we expected: the eigenvalue $2t^2$ has a 3-fold degeneracy consistent with $S=1$, $M=\pm 1, 0$ and the eigenvalue 0 is non-degenerate.

To find the combination of $|m_1, m_2\rangle$ which has spin $S=0=M$, note that

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has the null eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Thus the state must be $\frac{1}{\sqrt{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |-\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$

~~The remaining th~~ Hence:

$$|S=0, M=0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |-\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$$

$$|S=1, M=1\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|\;''\quad M=-1\rangle = |-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|\;''\quad M=0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle + |-\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$$

Note that $S=1$ states are symmetric under exchange of states, while $S=0$ is antisymmetric.

This could have been predicted, as follows. Define the permutation operator P by

$$P\,|m_1, m_2\rangle = |m_2, m_1\rangle$$

Now
$$P\,\vec{S}^{(1)} = \vec{S}^{(2)} P$$
$$P\,\vec{S}^{(2)} = \vec{S}^{(1)} P$$

Thus $P\vec{S} = \vec{S}P$ ie $[P, \vec{S}] = 0$

Hence $[P, \vec{S}^2] = 0$. So we can diagonalise P along with $\vec{S}^2$.

Clearly from $|m_1, m_2\rangle$ we get 3 even states $|\tfrac{1}{2}, \tfrac{1}{2}\rangle$, $|-\tfrac{1}{2}, -\tfrac{1}{2}\rangle$, $\tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |-\tfrac{1}{2}, \tfrac{1}{2}\rangle\right)$.

and one odd state $\tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |-\tfrac{1}{2}, \tfrac{1}{2}\rangle\right)$.

At the end, the change of basis is

$$\begin{array}{c} |S,M\rangle \\ \begin{pmatrix} |1,1\rangle \\ |1,0\rangle \\ |0,0\rangle \\ |1,-1\rangle \end{pmatrix} \end{array} = \left(\begin{array}{c|cc|c} 1 & & & \\ \hline & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \\ \hline & & & 1 \end{array}\right) \begin{array}{c} |m_1,m_2\rangle \\ \begin{pmatrix} |\frac{1}{2},\frac{1}{2}\rangle \\ |\frac{1}{2},-\frac{1}{2}\rangle \\ |-\frac{1}{2},\frac{1}{2}\rangle \\ |-\frac{1}{2},-\frac{1}{2}\rangle \end{pmatrix} \end{array}$$

$\downarrow$

Clebsch-Gordan coefficients. $\Rightarrow C_{s_1 m_1 s_2 m_2; S M}$

General case

Consider $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$.
These form a total of $(2j_1+1)$ and $(2j_2+1)$ states respectively.
Thus the combined system has the $(2j_1+1)(2j_2+1)$ states $|j_1, m_1; j_2, m_2\rangle$.

Here $m_1 = j_1, j_1-1, \cdots -j_1$
$m_2 = j_2, j_2-1, \cdots -j_2$.

Now we see that ~~the~~ possible values of m_1+m_2 are $j_1+j_2,\ j_1+j_2-1, \cdots, -(j_1+j_2)$
with degeneracies: $1,\ 2,\ 3, \cdots, 2, 1$.

As an example if $j_1 = \frac{3}{2}$, $j_2 = 2$ then we have

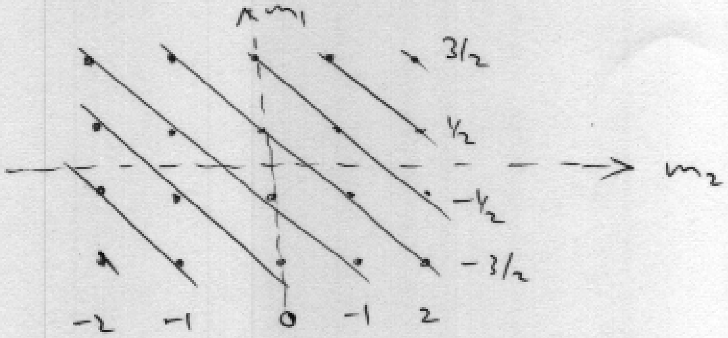
$$m_1+m_2 = M = \frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}$$

Deg: 1 2 3 4 4 3 2 1

Here the total number of states is 20
$= (2\cdot\frac{3}{2}+1)(2\cdot 4+1)$.

~~In general the sum $1+2+3+\cdots+2+1$ (with $2(j_1+j_2)+1$ terms) equals~~

In general, the degeneracies are computed as follows:



$M = m_1 + m_2$. Lines of fixed $m_1 + m_2$ are at $45°$.

Clearly every point in the square is counted once. So we get $(2j_1+1)(2j_2+1)$ states as expected.

~~what we now see is that they~~

~~are organised in representations~~ :

~~$J = j_1 + j_2$,~~

Now notice that the <u>highest</u> value of M $(= j_1 + j_2)$ must correspond to $J = j_1 + j_2$. This has a tower of $2(j_1+j_2)+1$ states as M goes from J to $-J$.

Once we remove those states we are left with states having a maximum $M = j_1 + j_2 - 1$.

~~These m~~

Since $M = 2$ is now singly degenerate, this ~~...~~ corresponds to a $J = j_1 + j_2 - 1$ rep ~~states~~, which has $2(j_1 + j_2 - 1) + 1$

$= 2(j_1 + j_2) - 1$ states in it.

Continuing in this way we end up with

$$
\begin{array}{ll}
J = j_1 + j_2 & 2(j_1 + j_2) + 1 \\
J = j_1 + j_2 - 1 & 2(j_1 + j_2) - 1 \\
J = j_1 + j_2 - 2 & 2(j_1 + j_2) - 3 \\
\vdots & \vdots
\end{array}
$$

This ends at $J = |j_1 - j_2|$ and one can check that

$$2(j_1 + j_2) + 1 + 2(j_1 + j_2) - 1 + \cdots + 2|j_1 - j_2| + 1$$

$= (2j_1 + 1)(2j_2 + 1)$ as one can check.

Thus we have completely unravelled the structure of $|j_1 m_1; j_2 m_2\rangle$ viewed ~~as~~ ~~eigenstates~~ in the basis $|j_1 j_2; J M\rangle$.

~~to find the~~ the latter basis has

$$J = j_1 + j_2,\ j_1 + j_2 - 1,\ \ldots\ |j_1 - j_2|$$

and $M = J, J-1, \ldots, -J$.

Now use the fact that for a single particle,

$$J_\pm |j, m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\ |j, m\pm 1\rangle$$

(Exercise).

We ~~can~~ now want an explicit formula:

$$|J, M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1, j_2; m_1, m_2 | J, M\rangle\ |j_1, j_2; m_1, m_2\rangle$$

(Note: $|J, M\rangle$ is also an eigenstate of $\vec{J}^{(1)2}$ and $\vec{J}^{(2)2}$ so it can also be labelled $|j_1, j_2; J, M\rangle$.)

Here $\langle j_1, j_2; m_1, m_2 | J, M\rangle$ are also written $C^{j_1 j_2 J}_{m_1 m_2 M}$ or $C_{j_1 m_1, j_2 m_2; JM}$.

There is no simple formula for C-G but they can be developed recursively using lowering operators.

First, note that $|J, M\rangle$ is easy to find for $J = j_1 + j_2$, $M = m_1 + m_2$. In this case it has to be just equal to $|j_1 j_2; m_1 m_2\rangle$ with $m_1 = j_1$, $m_2 = j_2$.

Hence:

$$|j_1 + j_2, j_1 + j_2\rangle = |j_1, j_2; m_1, m_2\rangle$$

Now act with J_-. We have

$$J_- |j_1 + j_2, j_1 + j_2\rangle = \hbar \sqrt{(j_1+j_2)(j_1+j_2+1) - (j_1+j_2)(j_1+j_2-1)}$$

$$\times |j_1 + j_2, j_1 + j_2 - 1\rangle$$

$$= \hbar \sqrt{2(j_1+j_2)}\, |j_1 + j_2, j_1 + j_2 - 1\rangle$$

On the RHS,

$$J_- |j_1, j_2; m_1, m_2\rangle = (J_-^{(1)} + J_-^{(2)})\, |j_1, j_2; j_1, j_2\rangle$$

$$\cancel{= \hbar(\sqrt{2j_1} + \sqrt{2j_2})\, |j_1, j_2;}$$

$$= \hbar\sqrt{2j_1}\, |j_1, j_2; j_1 - 1, j_2\rangle + \hbar\sqrt{2j_2}\, |j_1, j_2; j_1, j_2 - 1\rangle$$

Equating LHS and RHS, the factors of $\sqrt{2}\,\hbar$ cancel and we have:

$$|j_1+j_2, j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1, j_2; j_1-1, j_2\rangle$$
$$+ \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1, j_2; j_1, j_2-1\rangle$$

Continuing in this way we generate all $|J, M\rangle$ ~~of~~ the form $|j_1+j_2, M\rangle$.

Next we need to find $|j_1+j_2-1, j_1+j_2-1\rangle$.

Clearly this can only be a linear combination of $|j_1, j_2; j_1-1, j_2\rangle$ and $|j_1, j_2; j_1, j_2-1\rangle$. So we write

$$|j_1+j_2-1, j_1+j_2-1\rangle = \alpha\,|j_1, j_2; j_1-1, j_2\rangle$$
$$+ \beta\,|j_1, j_2; j_1, j_2-1\rangle$$

with $|\alpha|^2 + |\beta|^2 = 1$

Now this state has to be orthogonal to $|j_1+j_2, j_1+j_2-1\rangle$ which we have already found (top of page).

Thus $\alpha\sqrt{\frac{j_1}{j_1+j_2}} + \beta\sqrt{\frac{j_2}{j_1+j_2}} = 0$

or ~~$\alpha\sqrt{j_1}$~~ $\frac{\alpha}{\beta} = -\sqrt{\frac{j_2}{j_1}}$

$\therefore\ \alpha = \frac{\sqrt{j_2}}{\sqrt{j_1+j_2}}, \quad \beta = \frac{-\sqrt{j_1}}{\sqrt{j_1+j_2}}.$

Hence

$$|j_1+j_2-1, j_1+j_2-1\rangle = \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1, j_2; j_1-1, j_2\rangle$$

$$- \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1, j_2; j_1, j_2-1\rangle$$

Again we can use J_- to generate all the states $|j_1+j_2-1, M\rangle$ with $M = j_1+j_2-1, \ldots, -(j_1+j_2-1)$.

Next find the state $|j_1+j_2-2, j_1+j_2-2\rangle$. this must be a combination of:

$|j_1, j_2; j_1-2, j_2\rangle$, $|j_1, j_2; j_1-1, j_2-1\rangle$, $|j_1, j_2; j_1, j_2-2\rangle$

(Note that the first or last vectors could vanish, if j_1 or $j_2 \leq \frac{1}{2}$)

This vector has to be orthogonal to both $|j_1+j_2, j_1+j_2-2\rangle$ and $|j_1+j_2-1, j_1+j_2-2\rangle$. Together with normalisation, this fixes it completely.

Note that $\langle j_1, j_2; m_1, m_2 | J, M\rangle \neq 0$ only if: $m_1+m_2 = M$ <u>and</u>

$|j_1-j_2| \leq J \leq (j_1+j_2)$.