TUESDAY 15/2/11

The Hamiltonian of a spinless charged particle in a magnetic field is:

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\varphi(\vec{x}, t)$$

This leads to the Lorentz force law. To see this, notice that

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \Rightarrow \dot{x}^i = \frac{1}{m}(p_i - qA_i)$$

i.e. $p_i = m\dot{x}_i + qA_i(\vec{x}, t)$

In this situation, $F_i = \dot{p}_i = m\ddot{x}_i + q\frac{dA_i}{dt}$

$$\cancel{+q\frac{dA_i}{dt}} = m\ddot{x}_i + q\partial_t A_i + q\,\partial_j A_i\,\dot{x}_j$$

The Hamilton eqn is

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -\frac{1}{m}(p_j - qA_j).(-q\,\partial_i A_j) - q\partial_i\varphi$$

$$= q v_j \partial_i A_j - q\partial_i \varphi$$

Now $E_i = -\partial_t A_i - \partial_i \varphi$

Hence

$$m\ddot{x}_i + \cancel{q\partial_t A_i} = q v_j(\partial_i A_j - \partial_j A_i) + qE_i + \cancel{q\partial_t A_i}$$

$$\Rightarrow \boxed{m\ddot{x}_i = q(\vec{v} \times \vec{B})_i + qE_i}$$

Gauge invariance

Note that with $H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\varphi$,

the Schrodinger eqn:

$$H\psi = i\hbar \frac{\partial \psi}{\partial t}$$

is invariant under:

$$\varphi \to \varphi + \dot{\chi}$$
$$\vec{A} \to \vec{A} - \vec{\nabla}\chi$$
$$\psi \to e^{-i\frac{q}{\hbar}\chi}\psi$$

Check: $0 = H\psi - i\hbar\frac{\partial\psi}{\partial t}$

$$= \frac{1}{2m}(\cancel{p} - i\hbar\vec{\nabla} - q\vec{A})^2\psi$$
$$+ \left(q\varphi - i\hbar\frac{\partial}{\partial t}\right)\psi$$

Now under $\psi \to e^{-i\frac{q}{\hbar}\chi}\psi$,

$$-i\hbar\vec{\nabla}\psi \to e^{-i\frac{q}{\hbar}\chi}\big(-q\vec{\nabla}\chi\,\psi \; \cancel{\quad}$$
$$-i\hbar\vec{\nabla}\psi\big)$$

while $-q\vec{A}\psi \to e^{-i\frac{q}{\hbar}\chi}(-q\vec{A}\psi + q\vec{\nabla}\chi\,\psi)$

Thus $(-i\hbar\vec{\nabla} - q\vec{A})\psi \to e^{-i\frac{q}{\hbar}\chi}(-i\hbar\vec{\nabla} - q\vec{A})\psi$

Similarly $\left(q\varphi - i\hbar\frac{\partial}{\partial t}\right)\psi \to$

$$\left(q\varphi - i\hbar\frac{\partial}{\partial t}\right)e^{-i\frac{q}{\hbar}\chi}\psi$$

$$= e^{-i\frac{q}{\hbar}\chi}\Big[\big(q\varphi + q\dot{\chi} - q\dot{\chi}\;\cancel{\quad} - i\hbar\frac{\partial}{\partial t}\psi$$
$$e^{-iq/\hbar\,\chi}\left[q\varphi - i\hbar\frac{\partial}{\partial t}\right]\psi$$

Now consider a time-independent and uniform $\vec{B}$ magnetic field. The eqn of motion:

$$m\ddot{\vec{r}} = q\dot{\vec{r}} \times \vec{B}$$

can be solved by choosing $\vec{B} = (0,0,B)$.

Then,

$$m\ddot{x} = qB\dot{y}$$
$$m\ddot{y} = -qB\dot{x}$$
$$m\ddot{z} = 0$$

We can neglect the z-motion because it is just that of a free particle.

Now defining $\xi = x + iy$, we find:

$$\ddot{\xi} = -i\frac{qB}{m}\dot{\xi} = +i\omega\dot{\xi} \quad \text{where } \omega = -\frac{qB}{m}$$

(We will take $\omega < 0$, $qB > 0$).

The solution is: $\boxed{\xi = \xi_0 e^{+i\omega t} + \xi_1}$

where ξ_0, ξ_1 are arbitrary complex numbers.

The energy of this solution is:

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m|\dot{\xi}|^2 = \frac{1}{2}m\omega^2|\xi_0|^2$$

Now let us compute the angular momentum of the solution.

$$L_z = xp_y - yp_x = x(m\dot{y} + qA_y) - y(m\dot{x} + qA_x)$$
$$= m(x\dot{y} - y\dot{x}) + q(xA_y - yA_x)$$

(2A)

We will choose $A = \frac{1}{2} B(-y, +x)$ since this gives $(\vec{\nabla} \times \vec{A})_z = \partial_x A_y - \partial_y A_x = B$ as desired (however note that this is just one possible choice of gauge! Another choice is $A = B(0, x)$ or $A = B(-y, 0)$. There are infinitely many such choices.).

Then $L_z = m(x\dot{y} - y\dot{x}) + \frac{qB}{2}(x^2 + y^2)$.

The second term can be thought of as the angular momentum coming from the field, while the first term is the usual "mechanical" angular momentum.

In a moment we will see that only the sum is conserved - each term separately is time-dependent.

Using $\xi = x + iy = \xi_0 e^{i\omega t} + \xi_1$, we find:

$$L_z = \frac{m}{2i}(\bar{\xi}\dot{\xi} - \xi\dot{\bar{\xi}}) + \frac{qB}{2}\xi\bar{\xi}$$

$$= \frac{m}{2i}\Big([\bar{\xi}_0 e^{-i\omega t} + \bar{\xi}_1]\xi_0 e^{i\omega t} \cdot i\omega$$
$$- [\xi_0 e^{i\omega t} + \xi_1]\bar{\xi}_0 e^{-i\omega t} \cdot -i\omega$$
$$+ \frac{qB}{2}|\xi_0 e^{i\omega t} + \xi_1|^2 \Big)$$

$$\doteq m\omega|\xi_0|^2 + \frac{m\omega}{2}\left(\xi_0\bar{\xi}_1 e^{i\omega t} + \text{c.c.}\right)$$

$$+ \frac{qB}{2}\left(|\xi_0|^2 + |\xi_1|^2 + \xi_0\bar{\xi}_1 e^{i\omega t} + cc\right)$$

Now $-\frac{qB}{m} = \omega$, so above =

$$= m\omega|\xi_0|^2 + \frac{m\omega}{2}\left(\xi_0\bar{\xi}_1 e^{i\omega t} + cc\right)$$

$$- \frac{m\omega}{2}\left(|\xi_0|^2 + |\xi_1|^2 + \xi_0\bar{\xi}_1 e^{i\omega t} + cc\right).$$

$$= \frac{m\omega}{2}\left(|\xi_0|^2 - |\xi_1|^2\right)$$

We see that the time dependent terms just cancelled out!

Note that the above expression for L_z is valid only in the particular gauge that we chose. In this gauge, there is a continuous range of angular momenta for fixed energy. The energy is determined by m, ω, $|\xi_0|$. But the angular momentum ~~can vary from 0 (if~~ ~~$|\xi_1| - |\xi_0|$) to $-\frac{m\omega}{2}|\xi_0|^2$~~ takes any arbitrary negative value (if $|\xi_1|$ is large enough) with no change in the energy. This is not too surprising since the problem is translation invariant – but angular momentum is defined around a fixed origin.

Now consider the quantum mechanical problem. Here we have:

$$H = \frac{1}{2m}\left(\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}})\right)^2$$

and we may again choose the gauge

$$\hat{\vec{A}} = \frac{1}{2}B(-\hat{y}, \hat{x})$$

We would like to solve:

$$H\psi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t}\psi(\vec{r}, t)$$

The commutation relations among x, p are standard (as they must be):

$$[x_i, x_j] = [p_i, p_j] = 0$$

$$[x_i, p_j] = i\hbar\delta_{ij}$$

However, the velocities do not commute!

$$\begin{aligned}
[v_i, v_j] &= \frac{1}{m^2}[p_i - qA_i, p_j - qA_j] \\
&= -\frac{q}{m^2}[A_i, p_j] - \frac{q}{m^2}[p_i, A_j] \\
&= -\frac{q}{m^2}\left([A_i, -i\hbar\partial_j] + [-i\hbar\partial_i, A_j]\right) \\
&= \frac{i\hbar q}{m^2}(\partial_i A_j - \partial_j A_i) \\
&= \frac{i\hbar q}{m^2}\epsilon_{ijk}B_k
\end{aligned}$$

(for an arbitrary direction of B_i)

Now ~~consider~~ return to the case of a uniform (z-direction) magnetic field: $\vec{B} = (0, 0, B)$.

Then:
$$[v_x, v_y] = \frac{i\hbar q}{m^2} B = -\frac{i\hbar}{m}\omega$$
$$[v_y, v_z] = [v_x, v_z] = 0.$$

Now notice that the Hamiltonian can be split as:
$$H = \frac{m}{2}\vec{v}^2 = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2)$$
$$= H_\perp + H_\parallel$$

where
$$H_\perp = \frac{m}{2}(v_x^2 + v_y^2)$$
$$H_\parallel = \frac{m}{2}v_z^2$$

and clearly $[H_\perp, H_\parallel] = 0$.

Hence we can look for simultaneous eigenvectors of $H_\perp, H_\parallel$.

For $H_\parallel$ we just have a free particle: the wave fns are $e^{ik_z z}$ with $\hbar k_z = m v_z$, and k_z (or v_z) are arbitrary.

Now notice that the Hamiltonian ~~for~~ $H_\parallel$ has the form of a 1d harmonic oscillator:

$$H_\parallel = \frac{m}{2}(v_x^2 + v_y^2) \quad \text{with} \quad [v_x, v_y] = -\frac{i\hbar}{m}\omega$$

Let's rescale: $V_y = v_y\sqrt{\frac{m}{\hbar|\omega|}}$

$V_x = v_x\sqrt{\frac{m}{\hbar|\omega|}}$

Assume $\omega < 0$
$\therefore qB > 0$.

Then $H_{\perp} = \frac{\hbar\omega}{2}\left(V_x^2 + V_y^2 \right)$

with $[V_x, V_y] = i$ $\Rightarrow$ analogous to 1d SHO!

~~The degeneracy of~~

then the allowed energy eigenvalues are

$$E_{\perp} = \frac{\hbar\omega}{2}\left(n+\tfrac{1}{2}\right)$$

added to an arbitrary $E_{\parallel} = \frac{m}{2} V_z^2$ with an arbitrary value of V_z.

However the issue of <u>degeneracy</u> is subtle.

In the position basis,

$$V_x = \sqrt{\frac{m}{\hbar\omega}}\, v_x = \sqrt{\frac{m}{\hbar\omega}} \cdot \frac{1}{m}\left(p_x - qA_x\right)$$

$$\longrightarrow \frac{1}{\sqrt{m\hbar\omega}}\left(-i\hbar\frac{\partial}{\partial x} - qA_x(\vec{r})\right)$$

and $V_y = \dfrac{1}{\sqrt{m\hbar\omega}}\left(-i\hbar\dfrac{\partial}{\partial y} - qA_y(\vec{r})\right)$

Therefore the "ground state" (lowest Landau level) is a wave fn satisfying:

$$(V_x + iV_y)|0\rangle = 0$$

$$\Rightarrow \left[-i\hbar\left(\partial_x + i\partial_y\right) - q\left(A_x + iA_y\right)\right]\psi_0(x,y) = 0$$

Defining $\xi = x+iy$, we have $\left(\begin{array}{l} x = \frac{1}{2}(\xi+\bar{\xi}) \\ y = \frac{1}{2i}(\xi-\bar{\xi}) \end{array}\right)$

$$\frac{\partial}{\partial\xi} = \frac{\partial x}{\partial\xi}\partial_x + \frac{\partial y}{\partial\xi}\partial_y = \frac{1}{2}\left(\partial_x - i\partial_y\right)$$

Thus $\partial_x - i\partial_y = 2\partial_\xi$

$\partial_x + i\partial_y = 2\partial_{\bar\xi}$

Hence $\psi_0(\xi,\bar\xi)$ satisfies:

$$\left(-i\hbar\, 2\partial_{\bar\xi} - q.2A_{\bar\xi}\right)\psi_0(\xi,\bar\xi) = 0$$

ie:
$$\partial_{\bar\xi}\psi_0(\xi,\bar\xi) = \frac{iq}{\hbar} A_{\bar\xi}\psi_0(\xi,\bar\xi)$$

$$\Rightarrow \psi_0(\xi,\bar\xi) = f(\xi)\, e^{i\frac{q}{\hbar}\int^{\bar\xi} A_{\bar\xi}\, d\bar\xi'}$$

where $f(\xi)$ is arbitrary!. ~~The remaining~~ We see that the lowest Landau level is infinitely degenerate. As a consequence, all levels (defined by

$$\psi_n(x,y) \sim (V_x - iV_y)^n\,\psi_0(x,y)$$)

are infinitely degenerate.

This appear to pose a puzzle, which is resolved when we realise that the original problem is unphysical: a constant magnetic field in the z direction for all x & y!.

We did not take account of the finite size of the apparatus.

To make things more explicit let's consider the actual value of $\vec{A}$.

We have $\vec{\nabla} \times \vec{A} = \vec{B} =$ constant,

so a possible choice is

$$\vec{A} = \tfrac{1}{2} B(-y, x)$$

With this choice, $A_{\bar{\xi}} = \frac{1}{2}(A_x + iA_y)$

$$= \tfrac{1}{4} B(-y + ix)$$

$$= \frac{iB}{4}(x+iy) = \frac{iB}{4}\xi$$

Hence $\int A_{\bar{\xi}}\, d\bar{\xi} = \frac{iB}{4} \bar{\xi}\xi$

With ~~this~~, the ground state wave function becomes

$$\psi_0(\xi, \bar{\xi}) = f(\xi)\, e^{-\frac{qB}{4\hbar}\xi\bar{\xi}}$$

We see that it is normalisable, as desired, as long as $f(\xi)$ is a polynomial (of finite degree) in ξ.

Notice now that $\frac{\hbar}{qB}$ has dimensions of length. Let's call it ℓ^2.

Then $\psi_0(\xi, \bar{\xi}) \sim e^{-\frac{1}{4}\frac{\xi\bar{\xi}}{\ell^2}}$

Now a basis for the ground state wave fns $f(\xi)e^{-\frac{1}{4}\xi\bar{\xi}/\ell^2}$ is

$$\psi_0^{(m)}(\xi, \bar{\xi}) = \xi^m e^{-\frac{1}{4}\xi\bar{\xi}/\ell^2}$$

To gain physical insight, we notice that the angular momentum in the plane is $-i\frac{\partial}{\partial\theta}$ where

$$\xi = |\xi| e^{i\theta} .$$

Now $\psi_0^{(m)}(\xi, \bar{\xi}) \simeq \xi^m e^{-\xi\bar{\xi}/4\ell^2}$ are eigenfns of L_z with eigenvalue $-m$ (which, like the classical angular momentum, can go to $-\infty$!

Now we realise that if the system is in a finite box (of ~~size~~ area say $l_x \times l_y = A$) then the above analysis is valid only if $A \gg \ell^2$ (then near the walls of the box the wave fn would have died off fast enough).

We can estimate the size of the wave fn by considering

$$|\psi| = (\xi\bar{\xi})^m e^{-\xi\bar{\xi}/2\ell^2}$$

$$= \alpha^m e^{-\alpha/2\ell^2}$$

This ~~wave fn~~ probability has a maximum when

$$\frac{d}{d\alpha}\left(\alpha^m e^{-\alpha/2\ell^2}\right) = 0 \quad \rightarrow \quad \frac{m}{\alpha} = \frac{1}{2\ell^2}$$

ie $\frac{\alpha}{\ell^2} \sim m$.

Thus the degeneracy (maximum allowed ~~size of the~~ value of m) must be such that $A/\ell^2 \sim$ [illegible] where $\ell^2 = \hbar/qB$.

We conclude that the lowest Landau level has a degeneracy, in a finite system of area A, given (in order of magnitude) by $m_{max} \sim \frac{A}{l^2}$

where $l^2 = \frac{\hbar}{qB}$.

The "filled" lowest Landau level (here assume 1 fermion per state) is called a Hall droplet.