Lecture 19

In this lecture we will see how to compute scattering of a charged particle off a Coulomb potential.

For this we use the technique of time-dependent perturbation theory, by "switching on" the potential at a fixed time and using Fermi's Golden Rule to evaluate the transition rate from an incoming to scattered wave.

We have:

$$\frac{d}{dt}\Delta P_{\vec{p}_i \to \vec{p}_f} = \frac{2\pi}{\hbar}\int |\langle \vec{p}_f | \lambda W | \vec{p}_i \rangle|^2 \, \delta\left(\frac{\vec{p}_f^{\,2}}{2m} - \frac{\vec{p}_i^{\,2}}{2m}\right) p_f^2 \, dp_f \, d\Omega$$

$$= \frac{2\pi m p_i \lambda^2}{\hbar} |\langle \vec{p}_f | W | \vec{p}_i \rangle|^2 \, d\Omega$$

(here, $|\vec{p}_i| = |\vec{p}_f| = \hbar k$)

Now the initial state is Ae^{ikz}. The previous calculation assumes δ-function normalisation, so $A = \frac{1}{(2\pi\hbar)^{3/2}}$. The incident probability flux is $|A|^2 v_G$ as we have seen, with $v_G = \frac{\hbar k}{m}$. Thus,

$$d\sigma = \frac{1}{|A|^2 v_G} \cdot \frac{d}{dt} \Delta P_{\vec{p}_i \to \vec{p}_f}$$

$$= \frac{2\pi m p_i \lambda^2}{\hbar} \cdot \frac{m}{\hbar k} \frac{1}{|A|^2} |\langle \vec{p}_f | W | \vec{p}_i \rangle|^2 d\Omega$$

$$= \frac{2\pi m^2 \lambda^2}{\hbar |A|^2} \quad |\langle \vec{p}_f | W | \vec{p}_i \rangle|^2 \, d\Omega$$

where we used $p_i = \hbar k$.

Next, $\langle \vec{p}_f | W | \vec{p}_i \rangle = |A|^2 \int d^3\vec{r}\,' e^{-i(\vec{p}_f - \vec{p}_i)\cdot\vec{r}\,'} W(r')$

Combining and using $A = \frac{1}{(2\pi\hbar)^{3/2}}$, we finally get:

$$d\sigma = \left| \frac{m\lambda}{2\pi\hbar^2} \int e^{-i\vec{q}\cdot\vec{r}\,'} W(r') d^3\vec{r}\,' \right|^2 d\Omega$$

This is the Born Approximation. Here $\hbar\vec{q} = \vec{p}_f - \vec{p}_i$ is the momentum transfer.

The scattering amplitude is upto a phase just the square-root of $\frac{d\sigma}{d\Omega}$, so:

$$f_k(\theta) \sim \frac{m}{2\pi\hbar^2} \int e^{-i\vec{q}\cdot\vec{r}'} V(\vec{r}')\, d^3r'$$

ie in the Born approx, the scattering amplitude is the Fourier transform of the potential w.r.t. the momentum transfer.

We can now see explicitly that f_k is independent of φ if $V = V(|\vec{r}|)$:

$$f_k(\theta) \sim \frac{m}{2\pi\hbar^2} \int r'^2 dr'\, d(\cos\theta')\, d\varphi'\, e^{-iqr'\cos\theta'}$$

$$= \frac{2m}{\hbar^2} \int_0^\infty \frac{\sin qr'}{q} \cdot V(r') r' dr' \quad \text{where } q = \frac{\sqrt{|\vec{p}_f - \vec{p}_i|^2}}{\hbar}$$

$$= \frac{1}{\hbar} \hbar\sqrt{(2k^2 - 2k^2\cos\theta)}$$

$$= 2k\sin\theta/2$$

We also see that for $V(r') \sim \frac{1}{r'}$, the scattering amplitude diverges, as already predicted.

However we can use the Yukawa potential.

$V(r) = g\frac{e^{-\mu r}}{r}$, which is physically relevant, gives good convergence and reduces to the Coulomb potential when $\mu \to 0$.

In this case, $f_k(\theta) \sim \frac{2m}{\hbar^2}\cdot\frac{g}{q}\cdot\int_0^\infty \sin qr' e^{-\mu r'} dr'$

The integral gives $\frac{q}{\mu^2+q^2}$, so $f_k(\theta) \sim \frac{2mg}{\hbar^2(\mu^2+q^2)}$.

Using $q = 2k\sin\theta/2$ we get:

$$\frac{d\sigma}{d\Omega} = \frac{4m^2g^2}{\hbar^4(\mu^2+4k^2\sin^2\theta/2)^2}$$ which as $\mu \to 0$ gives

the differential cross-section for Coulomb scattering:

$$\left.\frac{d\sigma}{d\Omega}\right|_{Coulomb} = \frac{m^2(Ze^2)^2}{4(\hbar k)^4\sin^4\theta/2} = \frac{m^2Z^2e^4}{4p^4\sin^4\theta/2}$$

where we used $g = Ze^2$ and $p = \hbar k$. Though this answer is in principle only approximate (1st order in perturbation theory), it turns out to be <u>exact</u> for the

Coulomb potential! In fact the same formula even arises classically: "Rutherford scattering".

Now let us discuss the validity of the Born approximation. Suppose the range of the potential is a. For example, the Yukawa potential has a range $a \sim \frac{1}{k}$. Then $\frac{d\sigma}{d\Omega}$ as obtained in the Born approximation should be $\ll a^2$. This gives:

$$\frac{m^2}{\hbar^4} \left| \int V e^{-i\vec{q}\cdot\vec{r}} d^3\vec{r} \right|^2 \ll a^2.$$

If we first ignore the phase in the integral then we can estimate LHS as:

$\sim \frac{m^2}{\hbar^4} (V \cdot a^3)^2$. Then, $V \ll \frac{\hbar^2}{m a^2}$.

The phase, however, can assist convergence. For large k it suppresses the integral by $\sim \frac{1}{ka}$. Thus, the requirement is $V \ll \frac{\hbar^2}{ma^2} \cdot ka$

This means that given a finite-range potential and a very fast-moving particle, it is always valid. For slow particles $ka \lesssim 1$, the approximation is only good if the potential is quite short-range.

For the Coulomb potential the range is infinite, so we replace a on the RHS by r, and on the LHS we write $V \sim \frac{\alpha}{r}$ $(\alpha \sim e^2)$. then:

$$\frac{\alpha}{r} \ll \frac{\hbar^2 k}{mr} \Rightarrow \frac{\alpha m}{\hbar^2 k} = \frac{\alpha}{\hbar v} \ll 1$$

where v is the classical velocity.