Lecture 20

Scattering of identical particles

So far, we have considered the scattering of an incident particle off a fixed target. Of course the same formulae apply if the target particle is not fixed. We just use the reduced mass of the pair (and work in the CM frame).

A new feature arises if the two particles are identical. In this case, Fermi/Bose statistics affect the scattering amplitude in a way that can be experimentally measured.

The basic idea is that with identical particles, the total wave function must be symmetric or antisymmetric, depending on whether they are bosons or fermions.

Now the total wave-fn is the product of a spin part and an orbital part. To know the symmetry of the orbital part, we first need to know that of the spin part.

For this we have to use some simple facts about total spin wave-fns.

We know that combing spin s with itself gives spins $2s, 2s-1, \ldots, 0$. Each of the total spin states arising in this way has a definite symmetry under exchange of the two particles.

As an example take $s = \frac{1}{2}$. Then we have:

$$\frac{1}{2} \otimes \frac{1}{2} = \underset{\text{sym}}{1} \oplus \underset{\text{antisym}}{0}$$

This is proved as follows. On the LHS we have $|\frac{1}{2}, \pm\frac{1}{2}\rangle \otimes |\frac{1}{2}, \pm\frac{1}{2}\rangle$ where the second label in each ket is the spin-projection m_s. Now if $m_s = \frac{1}{2}$ in both, then the total $M_s = +1$ and so this must be one of the states with total $S = 1$. Hence:

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle = |1, 1\rangle$$

(we are not worrying about normalisation here). The above state is clearly symmetric under exchange of the particles.

Now $|1,0\rangle$ is obtained from this by applying a lowering operator $\hat{S}^- = \hat{S}_1^- + \hat{S}_2^-$. This gives (upto normalisation):

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

which is also symmetric. One more $\hat{S}^-$ gives $|1,-1\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$. Thus we have proved that total spin 1 is symmetric.

It follows that total spin = 0 must be the antisymmetric combination

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle - |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

because this is orthogonal to the other three states. Hence we have proved our claim.

The general rule, which can be derived in the same way, is that the state obtained from $s \otimes s$ is symmetric for $2s, 2s-2, \ldots$ and antisymmetric for $2s-1, 2s-3, \ldots$

Now consider statistics. If s is half-integer then the particles are fermions and the total wave fn must be antisym under exchange.

Moreover, in this case the numbers $2s$, $2s-2$ ... are <u>odd</u> integers. These are the ones for which the spin wave fn is symmetric. It follows that the orbital wave fn must be antisymmetric. Thus we have shown that for fermions:

<u>odd</u> total spin $\rightarrow$ orbital wave fn is antisym.

Likewise <u>even</u> total spin $\rightarrow$ orbital wave fn is symmetric.

Now consider s = integer for each of the scattering particles. This means they are bosons and the total wave function must be symmetric. For the spin part, the numbers $2s$, $2s-2$ ... are now <u>even</u> integers and, as we saw, they correspond to symmetric spin wave fns. Therefore the orbital wave fn must be symmetric. Hence for bosons,

<u>even</u> total spin $\rightarrow$ orbital wave fn is symmetric.

and <u>odd</u> total spin $\rightarrow$ orbital wave fn is antisym.

But this is the <u>same</u> rule as for fermions!

Hence we have a universal rule for two identical particles of spin s:

Even total spin $\longrightarrow$ sym. orbital wave fn,
Odd total spin $\longrightarrow$ antisym. orbital wave fn.

Now the symmetry of an orbital wave fn is the same as its behaviour under $\theta \to \pi - \theta$, $\varphi \to \varphi + \pi$ which interchanges the two particles. Under this transformation, $\cos\theta \to -\cos\theta$ and:

$$P_\ell(\cos\theta) \longrightarrow P_\ell(-\cos\theta) = (-1)^\ell \; P_\ell(\cos\theta)$$

So for even total spin one only has even-ℓ partial waves and for odd total spin, odd-ℓ partial waves.

Now consider the scattering wave fns discussed earlier, which asymptotically behave as:

$$e^{ikz} + f_k(\theta)\, \frac{e^{ikr}}{r}$$

Under $\theta \to \pi - \theta$, this goes to:

$$e^{-ikz} + f_k(\pi - \theta)\, \frac{e^{ikr}}{r}$$

Therefore we should now consider wave fns that behave asymptotically like:

$$\Psi_\pm(\vec{r}) = (e^{ikz} \pm e^{-ikz}) + (f_k(\theta) \pm f_k(\pi-\theta))\,\frac{e^{ikr}}{r}$$

The cross-section will then be:

$$\frac{d\sigma}{d\Omega} = |f_k(\theta) \pm f_k(\pi-\theta)|^2$$

(even total spin $\rightarrow$ +
odd total spin $\rightarrow$ −)

There will now be <u>interference</u> terms

$$\sim \pm\left[f_k(\theta) f_k^*(\pi-\theta) + f_k^*(\theta) f_k(\pi-\theta)\right]$$

These arise purely from statistics but can be thought of as new "exchange interactions".

An important point to remember is that in going from $\frac{d\sigma}{d\Omega}$ to σ, one must integrate only over a solid angle of 2π rather than 4π. This is because of statistics: one half of the 4π solid angle is identified with the other by $\theta \to \pi-\theta$, $\varphi \to \varphi+\pi$ which maps a given 2-particle configuration to itself by exchanging the particles.