The Instanton Determinant

Here we discuss the evaluation of the determinant:

$$K = \left\{ \frac{\det\left(-m\frac{d^2}{dt^2} + V''(x_{cl}^1)\right)}{\det\left(-m\frac{d^2}{dt^2} + m\omega^2\right)} \right\}^{-1/2}$$

We will show that

$$K = \frac{\alpha}{\sqrt{\pi\hbar}}$$

where α appears in $\dot{x}_{cl} = \alpha e^{-\omega|t|}$ as $|t| \to \infty$. We have shown that $\alpha = 2a\omega$ for a double well, where

$$\omega^2 = \frac{V''}{m} \quad \text{at } x = \pm a .$$

To evaluate K, consider the eigenvalue equation:

$$-m\frac{d^2 y_n}{dt^2} + V''(x_{cl}^1)\, y_n = \lambda_n y_n$$

where x_{cl}^1 is the one-instanton solution. Recall that $m\ddot{x}_{cl}^1 = V'(x_{cl}^1)$.

Differentiating $\rightarrow$ $m\dddot{x}_{cl}^1 = V''(x_{cl}^1)\,\dot{x}_{cl}$

Hence

$$\left(-m\frac{d^2}{dt^2} + V''(x_{cl}^1)\right)\dot{x}_{cl}^1 = 0 .$$

In other words the lowest eigenfunction $y_0(t)$ is proportional to $\frac{dx_{cl}}{dt}$ and has eigenvalue 0! [This is approximate for finite τ, becoming exact only as $\tau \to \infty$]

There is a good reason why this happened. A time-translate of the instanton is another instanton (located at a different time). Therefore there is no cost in the action if we move the instanton by a small amount. This guarantees that the function $\frac{dx_{cl}}{dt}$ is (i) a ~~[illegible]~~ solution of the fluctuation equation and (ii) has eigenvalue 0. Such an ~~[illegible]~~ eigenfunction is called a "zero mode".

We should not integrate over the coefficient a_0 in

$$\eta(t) = \sum_n a_n y_n(t)$$

If we did, we would get infinity since the integrand does not depend on a_0!

Physics provides a way out: integrating over the zero mode for translation in time must give us the length τ of the time interval. So we exchange $\int da_0$ for $\int_{-\tau/2}^{\tau/2} dt$ by a change of variables.

For this we need the normalized eigenf. y_0 corresponding to $\frac{dx'_{cl}}{dt}$. We have:

$$\int_{-T/2}^{+T/2} \left(\frac{dx'_{cl}}{dt}\right)^2 dt = \int_{-a}^{+a} dx \frac{dx'_{cl}}{dt}$$

$$= \int_{-a}^{+a} dx \sqrt{2V} = S_1$$

Hence $y_0(t) = \frac{1}{\sqrt{S_1}} \frac{dx'_{cl}}{dt}$

Hence $dc_0 = \sqrt{S_1}\, dt$

It is useful to recall that the natural coefficient of the measure factor dc_n in the path integral was $\frac{1}{\sqrt{2\pi\hbar}}$. This means we write:

$$\frac{dc_0}{\sqrt{2\pi\hbar}} = \sqrt{\frac{S_1}{2\pi\hbar}}\, dt$$

Thereafter, we drop $\int dc_0$. The integral over dt was already done. So the only change is

$$K \longrightarrow \sqrt{\frac{S_1}{2\pi\hbar}}\, K'$$

where $K' = \left\{ \frac{\det'\left(-m\frac{d^2}{dt^2} + V''(x'_{cl})\right)}{\det\left(-m\frac{d^2}{dt^2} + m\omega^2\right)} \right\}^{-1/2}$

$\det'$ means det over only nonzero eigenvalues: $\prod_{n=1}^{\infty} \lambda_n^{-1/2}$.

~~Henceforth we drop the ' on t, and write~~

Thus $K' = \left\{ \dfrac{\det'(\cancel{+}m\frac{d^2}{dt^2} + V''(x'_{cl}))}{\det(\cancel{+}m\frac{d^2}{dt^2} + m\omega^2)} \right\}^{-1/2}$

Finally: how do we compute K? There is an elegant argument using complex analysis that provides a useful result.

Consider a general potential $W(t)$ and let's look at the eigenvalue problem:

$$\left[-m\frac{d^2}{dt^2} + W(t)\right]\psi = \lambda\psi$$

The boundary conditions are $\psi(-T/2) = \psi(T/2) = 0$. Let us consider all $\psi_\lambda(t)$ which satisfy only $\psi_\lambda(-T/2) = 0$. We can normalise ψ_λ so that $\dot{\psi}_\lambda(-T/2) = 1$. This function, as a function of λ, is an allowed eigenfunction of the problem only if $\psi_\lambda(T/2) = 0$.

By definition,

$$\det\left(-m\frac{d^2}{dt^2} + W(t)\right) = \prod_n \lambda_n$$

Now consider two different potentials, $W^{(1)}$ and $W^{(2)}$, ~~[illegible]~~ with their associated eigenfns ~~eigenvalues~~ $\psi^{(1)}_\lambda$, $\psi^{(2)}_\lambda$ and eigenvalues $\lambda^{(1)}_n$, $\lambda^{(2)}_n$.

Let's form the ratio

$$\frac{\det\left[-m\frac{d^2}{dt^2}+W^{(1)}-\lambda\right]}{\det\left[-m\frac{d^2}{dt^2}+W^{(2)}-\lambda\right]}$$

This vanishes, as a function of λ, if $\lambda=\lambda_n^{(1)}$ for each n. If the eigenvalues of $W^{(1)}$ are distinct then one gets a simple zero whenever $\lambda=\lambda_n^{(1)}$. By the same argument, one gets a simple pole when $\lambda=\lambda_n^{(2)}$. Hence the above determinant is proportional to $\prod_n \frac{(\lambda_n^{(1)}-\lambda)}{(\lambda_n^{(2)}-\lambda)}$. If $\lambda\to\infty$ in the complex plane the ratio of dets goes to 1, hence in fact the above ratio of determinants is exactly equal to $\prod_n \frac{(\lambda_n^{(1)}-\lambda)}{(\lambda_n^{(2)}-\lambda)}$.

Now consider functions $\psi_\lambda^{(1)}(t)$ that solve the diff. eqn

$$\left(-m\frac{d^2}{dt^2}+W^{(1)}(t)\right)\psi_\lambda^{(1)}(t)=\lambda\,\psi_\lambda^{(1)}(t)$$

for some λ, and also satisfy $\psi_\lambda^{(1)}(-T/2)=0$, $\dot\psi_\lambda(-T/2)=1$. These are not necessarily eigenfunctions of our problem because we haven't fixed $\psi_\lambda^{(1)}(T/2)$ to vanish.

Now consider the ratio $\dfrac{\psi_\lambda^{(1)}(\tau/2)}{\psi_\lambda^{(2)}(\tau/2)}$. When $\lambda = \lambda_n^{(1)}$ then $\psi_\lambda^{(1)}$ is a true eigenfunction, hence $\psi_\lambda^{(1)}(\tau/2) = 0$ whenever $\lambda = \lambda_n^{(1)}$. Likewise $\psi_\lambda^{(2)}(\tau/2) = 0$ whenever $\lambda = \lambda_n^{(2)}$. It follows that the above ratio is also equal to $\prod_n \dfrac{(\lambda_n^{(1)} - \lambda)}{(\lambda_n^{(2)} - \lambda)}$. Hence we have shown that, as fns of the complex variable λ,

$$\frac{\det\left(-m\frac{d^2}{dt^2} + W^{(1)} - \lambda\right)}{\det\left(-m\frac{d^2}{dt^2} + W^{(2)} - \lambda\right)} = \frac{\psi_\lambda^{(1)}(\tau/2)}{\psi_\lambda^{(2)}(\tau/2)}$$

Now set $\lambda = 0$, to get

$$\frac{\det\left(-m\frac{d^2}{dt^2} + W^{(1)}(t)\right)}{\det\left(-m\frac{d^2}{dt^2} + W^{(2)}(t)\right)} = \frac{\psi_0^{(1)}(\tau/2)}{\psi_0^{(2)}(\tau/2)}$$

Thus $\det\left(-m\frac{d^2}{dt^2} + W^{(1)}(t)\right) = \text{Const.}\ \psi_0^{(1)}(\tau/2)$.

As an example, for the harmonic oscillator: Now we have $W(t) = m\omega^2$ (constant).

$$-m\ddot{\psi}_0 + m\omega^2\psi_0 = 0 \Rightarrow$$

$$\psi_0(t) = Ae^{\omega t} + Be^{-\omega t}$$

Now $\psi_0(-\tau/2) = 0$, $\dot{x}(-\tau/2) = 1 \Rightarrow$

$$\psi_0(t) = \frac{1}{\omega}\sinh\omega(t + \tau/2)$$

Now $\psi(T/2) = \frac{1}{\omega} \sinh \omega T$

Thus $\left(-m\frac{d^2}{dt^2} + m\omega^2\right)^{-1/2} = \frac{\sqrt{\omega}}{\sqrt{\sinh \omega T}} \times$ constants.

We can now fill in the constants (indep of W) to be $\sqrt{\frac{m}{2\pi i\hbar}}$ (since RHS should be $\sqrt{\frac{m\omega}{2\pi i\hbar \sinh \omega t}}$

Hence in general,

$$\boxed{\det{}^{-\frac{1}{2}}\left(-m\frac{d^2}{dt^2} + W(t)\right) = \left[\frac{2\pi i\hbar}{m}\psi_0(T/2)\right]^{-\frac{1}{2}}}$$

This is almost what we need - but while it worked for the harmonic oscillator, it will not work when we have a zero mode. In this situation we want to calculate

$$\det{}'^{-1/2}\left(-m\frac{d^2}{dt^2} + W(t)\right)$$

which is more subtle. This is the case where ~~we already have~~ $W(t) = V''(x_{cl}(t))$. ~~[illegible]~~

So we consider the ~~[illegible]~~ equation:

$$\left[-m\frac{d^2}{dt^2} + V''(x^1_{cl}(t))\right]\psi_\lambda(t) = \lambda\psi_\lambda(t)$$

With $\lambda = 0$ we have a soln. (approximate for T finite)

$$x_0(t) = \frac{1}{\sqrt{S_0}}\frac{dx^1_{cl}}{dt}$$

and we know that $\dot{x}_{cl}(t) \to \alpha\, e^{-\omega|t|}$ ~~as~~ as $t \to \pm T/2$, where $\omega = \sqrt{\frac{V''(\pm a)}{m}}$, so

~~So, that $\dot{x}_{cl}(t) = \alpha e^{-\omega|t|}$ for large~~

$$x_0(t) \to \frac{\alpha}{\sqrt{S_1}} e^{-\omega t}$$

This does not yet give us $\Psi_0(t)$, which must vanish at $t = -T/2$. For this we look for another solution of

$$\left[-m\frac{d^2}{dt^2} + V''(x_{cl}(t)) \right] \Psi = 0$$

call it $y_0(t)$. ~~and norm~~ This will be an exponentially growing solution at large times:

$$y_0(t) = \pm \frac{\alpha}{\sqrt{S_1}} e^{\omega |t|} \quad \text{as } t \to \pm\infty.$$

now define

$$\Psi_0 \sim \left(e^{\omega T/2} x_0(t) + e^{-\omega T/2} y_0(t) \right)$$

As $t \to -T/2$, $x_0 \to \frac{\alpha}{\sqrt{S_1}} e^{\omega T/2}$ while $y_0 \to -\frac{\alpha}{\sqrt{S_1}} e^{\omega T/2}$, so $\Psi_0 \to 0$ as desired. Also,

$$\dot{\Psi}_0 \sim \left(e^{\omega T/2} \dot{x}_0 + e^{-\omega T/2} \dot{y}_0 \right)$$

$$\cancel{= A\left(-\omega e^{\omega T/2} \dot{x}_0 - \omega e^{\omega T/2} y_0\right.}$$

$$\to A\left(-\omega e^{\omega T/2} e^{-\omega T/2} - \omega e^{-\omega T/2} e^{\omega T/2}\right) \quad \text{where } A = \frac{\alpha}{\sqrt{S_1}}$$

$$= -2A\omega$$

Hence $\Psi_0(t) = \frac{1}{2A\omega} \left(e^{\omega T/2} x_0(t) + e^{-\omega T/2} y_0(t) \right)$

We want $\Psi_0(T/2)$, which is easily found to be:

$$\frac{A}{2A\omega}\left(e^{\omega T/2} e^{-\omega T/2} + e^{-\omega T/2} e^{\omega T/2} \right) = \frac{1}{\omega}$$

It follows that on a large but finite interval,

$$\frac{\det\left(-m\frac{d^2}{dt^2}+V''\right)}{\det\left(-m\frac{d^2}{dt^2}+m\omega^2\right)} = \frac{\frac{2\pi\hbar}{m\omega}}{\frac{2\pi\hbar}{m\omega}\sinh\omega T}$$

$$\rightarrow 2e^{-\omega T}$$

This vanishing as $T\rightarrow\infty$ indicates the appearance of the zero mode. To eliminate it, we must divide by the lowest eigenvalue λ_0 of the operator $-m\frac{d^2}{dt^2}+V''$. This can be estimated to be

$$\lambda_0 \simeq 4mA^2 e^{-\omega T}$$

(We will see this in an exercise). Here, A is the constant in

$$\frac{1}{\sqrt{S_0}}\frac{dx_{cl}}{dt} \rightarrow A e^{-\omega|t|} \quad \text{as } t\rightarrow\pm\infty.$$

So it is calculable for any given potential and its 1-instanton solution.

Hence
$$\frac{\det'\left(-m\frac{d^2}{dt^2}+V''\right)}{\det\left(-m\frac{d^2}{dt^2}+m\omega^2\right)} \rightarrow \frac{2e^{-\omega T}}{4mA^2 e^{-\omega T}} = \frac{1}{2mA^2}$$

Thus
$$K' = \left[\frac{\det'\left(-m\frac{d^2}{dt^2}+V''\right)}{\det\left(-m\frac{d^2}{dt^2}+\omega^2\right)}\right]^{-1/2} \rightarrow \sqrt{2m}\,A$$

and hence

$$K = \left(\frac{S_0}{2\pi\hbar}\right)^{1/2} K' = \left(\frac{m S_0}{\pi\hbar}\right)^{1/2} A = \sqrt{\frac{m}{\pi\hbar}}\,\alpha$$

now the WKB energy splitting formula was:

$$E_\pm = \frac{\hbar\omega}{2} \pm \hbar K e^{-S_1/\hbar}$$

$$= \frac{\hbar\omega}{2} \pm \sqrt{\frac{m\hbar}{\pi}}\,\alpha\, e^{-S_1/\hbar}$$

where α satisfies:

$$\dot{x}_{cl} \to \alpha e^{-\omega|t|} \quad \text{for } t \to \pm\infty .$$

As an example, if $V = \frac{\lambda}{4!}(x^2 - a^2)^2$ then

$$x_{cl} = a \tanh \sqrt{\frac{\lambda a^2}{12m}}\, t$$

$$= a \tanh \frac{\omega t}{2} \qquad \omega = \sqrt{\frac{\lambda a^2}{3m}}$$

so

$$\dot{x}_{cl} = \frac{\omega a}{2} \operatorname{sech}^2 \frac{\omega t}{2}$$

$$= \frac{\omega a}{2} \frac{1}{\cosh^2 \frac{\omega t}{2}}$$

$$\Rightarrow \frac{\omega a}{2} \cdot \frac{4}{\left(e^{\omega t/2} - e^{-\omega t/2}\right)^2} \to 2a\omega\, e^{-\omega t}$$

and hence $\alpha = 2a\omega = 2a^2\sqrt{\frac{\lambda}{3m}}$.

It only remains to show that the lowest eigenvalue λ_0 is:

$$\lambda_0 = 4mA^2 e^{-\omega T}$$

(as expected, $\lambda_0 \to 0$ as $T \to \infty$).

Recall that using $x_0(t)$, $y_0(t)$ which both solve

$$\left(-m \frac{d^2}{dt^2} + V''\right)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$$

we found $\psi_0 = \frac{1}{2A\omega}\left(e^{\omega T/2} x_0 + e^{-\omega T/2} y_0\right)$

which vanishes at $x = -T/2$ and has $\dot{\psi}_0 |_{t = -T/2} = 1$. Of course, being a linear combination of x_0 and y_0, ψ_0 also solves:

$$\left(-m \frac{d^2}{dt^2} + V''\right)\psi_0 = 0$$

Now we assume $\psi_{\lambda_0} = \psi_0 + \lambda_0 \psi_1$ and try to find ψ_1. For this we work to lowest order in λ_0 (since we expect it to be small). Thus:

$$\left(-m \frac{d^2}{dt^2} + V''\right)\left(\psi_0 + \lambda_0 \psi_1\right) = \lambda_0 \left(\psi_0 + \lambda_0 \psi_1\right) \simeq \lambda_0 \psi_0$$

Hence: $\left(-m\frac{d^2}{dt^2} + V''\right)\Psi_1 = \Psi_0$

which determines Ψ_1 in principle.

The solution of this, given two solutions $x_0(t)$, $y_0(t)$ of the homogeneous eqn, is standard:

$$\Psi_1 = -\frac{1}{2mA^2}\int_{-T/2}^{t} dt' \left(y_0(t)x_0(t') - x_0(t)y_0(t')\right)\Psi_0(t')$$

where $A = \frac{\alpha}{\sqrt{S_1}}$ has been introduced above.

To check that Ψ_1 solves the eqn at the top of the page, we need to use:

$$x_0\dot{y}_0 - y_0\dot{x}_0 = \text{constant}$$

(LHS = "Wronskian" which is easily shown to be a constant for the type of diff. eqn we are considering). We **fix** the constant by taking $t \to -\infty$ when $x_0 \sim Ae^{-\omega t}$ and $y_0 \sim Ae^{\omega t}$ so $\boxed{x_0\dot{y}_0 - y_0\dot{x}_0 = 2A^2}$

Hence we have:

$$\Psi_{\lambda_0}(t) = \Psi_0(t) - \frac{\lambda_0}{2mA^2}\int_{-T/2}^{t} dt' \left(y_0(t)x_0(t') - x_0(t)y_0(t')\right)\Psi_0(t')$$

Now λ_0 is fixed by requiring $\psi_{\lambda_0}(t) = 0$ at $t = T/2$. Thus:

$$\psi_{\lambda_0}(T/2) = 0 = \varphi_0(T/2) - \frac{\lambda_0}{2mA^2}\int_{-T/2}^{T/2} dt' \,(y_0(t)\, x_0(t') - x_0(t)\, y_0(t'))\, \psi_0(t')$$

$$\cancel{\varphi_0(T/2) - \frac{\lambda_0}{2mA^2}}$$

Now $\varphi_0 = \frac{1}{2A\omega}\left(e^{\omega T/2} x_0 + e^{-\omega T/2} y_0\right)$

Moreover x_0, y_0 are orthonormal ~~orthogonal~~ over the interval $-T/2$ to $T/2$. Thus the integral is:

$$\int_{-T/2}^{T/2} dt' \left(y_0\left(\tfrac{T}{2}\right) x_0(t') - x_0\left(\tfrac{T}{2}\right) y_0(t') \right)\left(\frac{1}{2A\omega}\right)\left(e^{\omega T/2} x_0(t') + e^{-\omega T/2} y_0(t')\right)$$

$$= \frac{A}{2A\omega}\int_{-T/2}^{T/2} dt' \left(e^{\omega T} x_0(t')^2 - e^{-\omega T} y_0(t')^2\right)$$

$$= \frac{1}{2\omega}\left(e^{\omega T} - e^{-\omega T}\right) \sim \frac{e^{\omega T}}{2\omega}$$

for large T. Also $\varphi_0(T/2) = 1/\omega$.

$$\psi_{\lambda_0}(T/2) = 0 = \frac{1}{\omega} - \frac{\lambda_0}{2mA^2}\cdot\frac{e^{\omega T}}{2\omega}$$

Hence $\lambda_0 = 4mA^2 e^{-\omega T}$ as desired!