Quantum Mechanics 2 – February-May 2011

Course instructor: Sunil Mukhi.
Course tutor: Nilay Kundu.

Home Assignment 3

Assigned: April 29, 2011, Due: May 10, 2011

1. Consider a time-independent Hamiltonian H_0 which has a spectrum of discrete eigenstates $|\psi_n\rangle = |\psi_1\rangle, \psi_2\rangle, \cdots$ with energy eigenvalues $E_n = E_1, E_2, \cdots$ as well as a spectrum of continuum eigenstates $|\psi_{E,\beta}\rangle$ with energies E, where β labels the degeneracy (for example it can correspond to the direction of the momentum $\vec{p}$ for free particles). Assume also that the discrete energies E_n are all > 0 and the continuous energies E range over all values from 0 to ∞. Thus, transitions can take place from a discrete state to a continuum state of the *same* energy.

   Now introduce a time-independent perturbation λW satisfying:

   $$\langle\psi_n|W|\psi_m\rangle = \langle\psi_{E,\beta}|W|\psi_{E',\beta'}\rangle = 0$$

   Hence the only non-vanishing matrix elements are $\langle\psi_{E,\beta}|W|\psi_m\rangle$ which can induce a transition from a discrete to a continuous state.

   (i) Show that in lowest-order perturbation theory, the probability for the state $|\psi_k\rangle$ to go to the continuum in a time t is:

   $$\mathcal{P}_{k,cont}(t) = \frac{1}{\hbar^2}\int_0^\infty dE\ K(E,k)\ \left(\frac{\sin(E-E_k)t/\hbar}{(E-E_k)/\hbar}\right)^2$$

   where

   $$K(E,k) \equiv \lambda^2 \int d\beta\ \rho(E,\beta)\,|\langle\psi_{E,\beta}|W|\psi_k\rangle|^2$$

   and $\rho(E,\beta)$ is the density of continuum states with energy E and other quantum numbers β.

   (ii) In the limit of large times, show that this tends to:

   $$\mathcal{P}_{k,cont}(t) = \frac{2\pi t}{\hbar}K(E=E_k,k)$$

   How large should the time be such that the above expression is valid? Express this in terms of some property of $K(E,k)$.

(iii) Defining the "decay constant":

$$\Gamma = \frac{d\mathcal{P}_{k,cont}}{dt} = \frac{2\pi}{\hbar} K(E = E_k, k)$$

show that lowest-order perturbation theory makes sense only if $t \ll \frac{1}{\Gamma}$.

(iv) Now consider a general wave function for $H_0 + \lambda W$ expanded over the stationary states of H_0:

$$|\psi(t)\rangle = \sum_n b_n(t)\, e^{-\frac{iE_n t}{\hbar}} |\psi_n\rangle + \int dE\, d\beta\, \rho(E,\beta)\ b_{E,\beta}(t)\, e^{-\frac{iEt}{\hbar}} |\psi_{E,\beta}\rangle$$

(the first term is very familiar while the second term is the analogous contribution from the continuous spectrum). Using the properties of W, show that:

$$\frac{d}{dt} b_k(t) = -\frac{1}{\hbar^2} \int_0^\infty dE \int_0^t dt'\, K(E,k)\, e^{\frac{i(E_k - E)(t-t')}{\hbar}}\ b_k(t') \qquad (*)$$

(v) Imposing the boundary condition that the system starts out in the state $|\psi_k\rangle$, namely:

$$b_k(t=0) = 1, \quad b_{E,\beta}(t=0) = 0$$

show that an approximate solution of the above equation is:

$$b_k(t) = e^{-\frac{\Gamma t}{2}} e^{-\frac{i\delta E t}{\hbar}}$$

(hint: approximate $b_k(t')$ by $b_k(t)$ on the RHS of Eq.($*$) and think of a reason why this is a good approximation). Hence find $\mathcal{P}_{kk}(t)$, the probability that the system has remained in its original state after time t, and thereby $\mathcal{P}_{k,cont}(t)$, the decay probability.

Closing remarks: You always "knew" that unstable states decay like $e^{-\Gamma t}$ where Γ is the decay constant, or $\tau = \frac{1}{\Gamma}$ is the lifetime. Here we have derived it in a very general situation. Observe that the system does not oscillate between ψ_k and $\psi_{E,\beta}$ unlike in the examples studied in class! The difference is crucially due to the fact that the latter state lies in the continuum rather than being a discrete state.

(10+10+10+35+35=100 marks)