Mini-Course on String Theory

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Contents:

1. Classical Bosonic Strings

(i) Review of particle mechanics

We are going to formulate and study the physics of relativistic superstrings.

How does this differ from the more traditional physics of relativistic point particles? For particles, we have two options:

(a) Relativistic particle mechanics (single particle, first quantised):

- Write the worldline action of a relativistic particle.
- It has reparametrisation invariance on the worldline.
- Gauge-fix this invariance.
- Quantise the particle and study free propagation.
- Introduce interactions and compute scattering amplitudes.

Amplitudes are computed in perturbation theory. The background in which the particle propagates is treated as external and fixed.

(b) Relativistic quantum field theory:

- Write down an interacting (nonlinear) action for classical fields.
- If the action has gauge symmetries, fix them.
- Quantise the fields.
- Compute scattering amplitudes.

Amplitudes can be computed in perturbation theory , but also non-perturbative effects may be studied, e.g. using semi-classical methods (solitons, instantons). The background in which the amplitudes are computed is obtained by solving the classical field equations.

Both approaches work only if the interactions are of renormalisable type. The second approach, however, is not tied to perturbation theory or fixed backgrounds. Therefore it is much more powerful.

In string theory, the analogue of the first approach (relativistic string mechanics) is rather well-understood, while the analogue of the second (relativistic string field theory) remains rather complicated and abstruse.

Therefore we will develop the first approach, by analogy with particle mechanics. This calls for a review of the relativistic point particle.

This is described by spacetime coordinates $X^\mu(t),\ \ \mu = 0, 1, \ldots, D-1$, along with an auxiliary variable $g(t)$. The worldline action is:

$$S = \frac{1}{2}\int dt\,\left(\frac{1}{\sqrt{g}}\,\dot{X}^\mu \dot{X}_\mu - m^2\sqrt{g}\right)$$

Under reparametrisations $t \to t'(t)$, we declare that

$$\begin{aligned} X^\mu(t) \ &\to\ X'^\mu(t') = X^\mu(t) \\ g(t) \ &\to\ g'(t') = \left(\frac{dt}{dt'}\right)^2 g(t) \end{aligned}$$

Then the action is invariant under reparametrisations of the worldline.

The equations of motion are:

$$\frac{\delta S}{\delta g(t)} = 0 \;\Rightarrow\; g^{-1}\dot{X}^\mu \dot{X}_\mu + m^2 = 0$$

$$\frac{\delta S}{\delta X^\mu(t)} = 0 \;\Rightarrow\; \frac{d}{dt}\left(\frac{1}{\sqrt{g}}\dot{X}_\mu\right) = 0$$

The canonical momentum is:

$$p_\mu = \frac{1}{\sqrt{g}}\dot{X}_\mu$$

The $g(t)$ equation of motion then says that

$$p_\mu p^\mu = -m^2$$

which means the particle has mass m.

The $X^\mu(t)$ equation tells us that

$$\dot{p}_\mu = 0$$

so the momentum is conserved.

From the first equation we can solve for $g(t)$:

$$g(t) = \frac{-\dot{X}^\mu \dot{X}_\mu}{m^2}$$

Substituting into the action we find:

$$S = -m \int dt \sqrt{-\dot{X}^\mu \dot{X}_\mu}$$

This action is the invariant length of the particle's trajectory. We see that reparametrisations of t are still a symmetry.

To quantise this theory, it will be necessary to make a gauge choice. A convenient choice is

$$X^0(t) = t$$

This fixes the reparametrisation invariance and leads to

$$S = -m \int dt \sqrt{1 - \dot{X}_i \dot{X}_i} \sim -m + \frac{1}{2} m \int dt \, (\dot{X}_i)^2, \qquad \dot{X}_i \ll 1$$

In this gauge, the momentum satisfying the equations of motion:

$$\dot{p}_\mu = 0$$

takes the familiar form:

$$p_0 = -\frac{m}{\sqrt{1-(\dot{X}_i)^2}}$$

$$p_i = \frac{m\dot{X}_i}{\sqrt{1-(\dot{X}_i)^2}}$$

We see that the gauge choice $X^0(t) = t$, while it breaks manifest Lorentz invariance, is a good one to provide physical insight.

An alternative approach is to not eliminate $g(t)$. Instead, we gauge-fix by using the worldline reparametrisation invariance to set $g(t) = 1$.

Unlike the previous gauge, this one does not break Lorentz invariance, and is called a covariant gauge. The action in this gauge is

$$S = \frac{1}{2} \int dt \, \left(\dot{X}^\mu(t) \dot{X}_\mu(t) - m^2 \right)$$

The action is simple, but we must keep track of the equations of motion that followed from the original action. These reduce to:

$$\begin{aligned} \dot{X}^\mu \dot{X}_\mu + m^2 &= 0 \\ \ddot{X}_\mu &= 0 \end{aligned}$$

The second of these equations follows from the gauge-fixed action, but the first one does not. We must impose it as a constraint.

Thus the velocities $\dot{X}^\mu(t)$ are not all independent.

The second equation is solved by writing

$$X^\mu(t) = X_0^\mu + p^\mu t$$

Thus the complete solution of the the classical relativistic free particle is given by solutions of the above equation, subject to the constraint.

This constraint is the price we pay for choosing a covariant gauge. However, there are two important benefits:

- The action is quadratic, with no square roots.
- The action and constraints both have a sensible $m \to 0$ limit.

(ii) The closed bosonic string

Proceeding by analogy, we define a string through its spacetime coordinates $X^\mu(\sigma,t)$ as well as an auxiliary worldsheet field $g_{ab}(\sigma,t)$, the worldsheet metric.

Here σ is a coordinate along the string, whose range is $0 \le \sigma \le \pi$, while $a, b = 0, 1$ label the worldsheet directions: $0 \leftrightarrow t,\ 1 \leftrightarrow \sigma$.

For the string to be closed, we require that

$$X^\mu(\sigma+\pi,t) = X^\mu(\sigma,t)$$

The action is chosen to be:

$$S = -\frac{T}{2}\int d\sigma\, dt\, \sqrt{-\|g\|}\,\Big(g^{ab}\,\partial_a X^\mu \partial_b X_\mu + \Lambda\Big)$$

where g^{ab} is the inverse and $\|g\|$ is the determinant of the metric.

The constant T has dimensions of length^{-2} which is the same (when $\hbar = c = 1$) as mass/length. It is called the string tension.

Λ is the worldsheet cosmological constant.

This action has reparametrisation invariance in two dimensions:

$$
\begin{aligned}
t &\longrightarrow t'(\sigma, t) \\
\sigma &\longrightarrow \sigma'(\sigma, t) \\
X^\mu(\sigma, t) &\longrightarrow X'^\mu(\sigma', t') = X^\mu(\sigma, t) \\
g_{ab}(\sigma, t) &\longrightarrow g'_{ab}(\sigma', t') = \frac{\partial \xi^c}{\partial \xi'^a}\frac{\partial \xi^d}{\partial \xi'^b}\, g_{cd}(\sigma, t)
\end{aligned}
$$

where $(\xi^0, \xi^1) = (t, \sigma)$.

The action is very similar to the point particle action that we wrote earlier.

Note that with a metric on the worldsheet, our theory is rather like two-dimensional gravity. However, there is no kinetic term for the metric.

In particular, the Einstein term $\int \sqrt{-\|g\|}\, R$ is a total derivative in two dimensions and does not lead to propagation of the metric field. The string action could be thought of as non-dynamical worldsheet gravity coupled to matter.

We will henceforth set $\Lambda = 0$, analogous to the case of a massless particle.

On doing this, we find that the action develops a new invariance, under:

$$g_{ab}(\sigma, t) \rightarrow e^{\rho(\sigma,t)}\, g_{ab}(\sigma, t)$$

with no change in worldsheet or spacetime coordinates. This is called Weyl invariance.

The equations of motion are straightforward to write down:

$$\frac{\delta S}{\delta g^{ab}(\sigma, t)} = 0 \;\Rightarrow\; \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} g_{ab}\, g^{cd}\, \partial_c X^\mu \partial_d X_\mu = 0$$

$$\frac{\delta S}{\delta X^\mu(\sigma, t)} = 0 \;\Rightarrow\; \partial_a \left(\sqrt{-\|g\|}\, g^{ab}\, \partial_b X_\mu \right) = 0$$

The first equation states the vanishing of the energy-momentum tensor:

$$T_{ab} \equiv -\frac{2}{T} \frac{1}{\sqrt{-\|g\|}} \frac{\delta S}{\delta g^{ab}} = 0$$

From reparametrisation invariance, this tensor is conserved: $\partial^a T_{ab} = 0$.

The equation $T_{ab} = 0$ is solved by writing:

$$g_{ab} = e^{\phi(\sigma,t)}\, \partial_a X^\mu \partial_b X_\mu$$

This is a solution for arbitrary $\phi(\sigma, t)$, because of Weyl invariance.

Inserting this into the action we get:

$$S = -T \int d\sigma\, dt\, \sqrt{-\|\partial_a X^\mu \partial_b X_\mu\|}$$

This action is equal to the invariant area of the worldsheet. It is still invariant under reparametrisations of σ, t as one can easily check. However, it is highly nonlinear in the string coordinates.

Now it is time to fix a gauge. By analogy with the point particle, we could choose:

$$X^0(\sigma,t) = t, \qquad X^1(\sigma,t) = \sigma$$

which is called the static gauge. The action remains nonlinear in the remaining X^i, and is hard to quantise.

A more convenient gauge is the one analogous to $g(t) = 1$ for a point particle. Thus, we again do not eliminate g_{ab} from the equations of motion, but instead fix two of its three independent components using reparametrisation invariance:

$$g_{ab} = e^{\phi(\sigma,t)}\,\eta_{ab}$$

This is called the conformal gauge. This is a covariant gauge since it preserves Lorentz invariance.

Because of Weyl invariance, the function $\phi(\sigma, t)$ decouples from the action, which becomes very simple:

$$S = -\frac{T}{2} \int d\sigma \, dt \, \partial_a X^\mu \partial^a X_\mu$$

In conformal gauge, the energy-momentum tensor can be re-derived as the conserved current for worldsheet translation invariance:

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} \eta_{ab} \, \eta^{cd} \, \partial_c X^\mu \partial_d X_\mu, \qquad \partial^a T_{ab} = 0$$

As before, after choosing a gauge we must make sure that all the original equations of motion are satisfied. The g_{ab} equation of motion is

$$T_{ab} = 0$$

and has to be implemented as a constraint.

This simplifies if we use light-cone coordinates on the worldsheet:

$$\xi^{\pm} = t \pm \sigma$$

$$\partial_{\pm} = \partial/\partial\xi^{\pm} = \frac{1}{2}(\partial_t \pm \partial_\sigma)$$

We find that the equation

$$T_{+-} = 0$$

is identically satisfied, while the remaining equations are:

$$T_{++} = \partial_+ X^\mu \, \partial_+ X_\mu = 0$$

$$T_{--} = \partial_- X^\mu \, \partial_- X_\mu = 0$$

These are called Virasoro constraints.

The X^μ equation of motion in conformal gauge becomes:

$$\partial_a \partial^a X^\mu \sim \partial_- \partial_+ X^\mu = 0$$

This is just the two-dimensional Klein-Gordon equation following from the conformal gauge action.

In particular, this implies that the Virasoro constraints satisfy:

$$\partial_- T_{++} = \partial_+ T_{--} = 0$$

which is the conservation equation in these coordinates.

The Klein-Gordon equation is solved by writing:

$$X^\mu(\sigma, t) = X_L^\mu(t + \sigma) + X_R^\mu(t - \sigma)$$

where X_L, X_R are arbitrary functions of one argument.

This essentially solves the classical bosonic closed string. The string coordinates are arbitrary functions of $t + \sigma$ or $t - \sigma$, subject to the requirements of periodicity of X^μ in σ and the Virasoro constraints.

We can solve the periodicity requirement by making a mode expansion.

First, it is convenient to define a new dimensional parameter α' by:

$$T = \frac{1}{2\pi\alpha'}$$

Now, an arbitrary periodic function of σ can be expanded in modes:

$$X_L^\mu(t+\sigma) = \frac{1}{2}x^\mu + \alpha'\, p^\mu\,(t+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}_n^\mu\, e^{-2in(t+\sigma)}$$

$$X_R^\mu(t-\sigma) = \frac{1}{2}x^\mu + \alpha'\, p^\mu\,(t-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\, e^{-2in(t-\sigma)}$$

Because of the linear term in σ, X_L and X_R are not separately periodic in σ, but their sum is periodic.

Reality of X_L and X_R imposes the requirement that:

$$(\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu, \qquad (\alpha_n^\mu)^* = \alpha_{-n}^\mu$$

In terms of these modes, we have:

$$\partial_+ X^\mu = \partial_+ X^\mu_L = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \tilde{\alpha}^\mu_n \, e^{-2in(t+\sigma)}$$

$$\partial_- X^\mu = \partial_- X^\mu_R = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \alpha^\mu_n \, e^{-2in(t-\sigma)}$$

where we have defined $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$.

Let us also expand the energy-momentum tensor in terms of modes:

$$T_{++} = 4\alpha' \sum_{n=-\infty}^{\infty} \widetilde{L}_n e^{-2in(t+\sigma)}$$

$$T_{--} = 4\alpha' \sum_{n=-\infty}^{\infty} L_n e^{-2in(t-\sigma)}$$

Then the Virasoro constraints amount to:

$$L_n = \widetilde{L}_n = 0, \quad n \in \mathbb{Z}$$

It is clear that the bilinear relations

$$T_{\pm\pm} = \partial_\pm X^\mu \, \partial_\pm X_\mu \equiv \partial_\pm X \cdot \partial_\pm X$$

impose a bilinear relation between L_n and α_n^μ, and similarly for the left movers.

This relation is easily seen to be:

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{-m} \cdot \alpha_{n+m}, \qquad \widetilde{L}_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \widetilde{\alpha}_{-m} \cdot \widetilde{\alpha}_{n+m}$$

Thus, in terms of Fourier modes, the Virasoro constraints that must be satisfied by the closed string coordinates are expressed as:

$$\widetilde{L}_n = L_n = 0$$

with $\widetilde{L}_n, L_n$ given as above.

(iii) The open bosonic string

For open strings, $X^\mu(\sigma, t)$ is no longer periodic in σ. Instead, the string must "end" at $\sigma = 0, \pi$.

At each end, we need to specify boundary conditions for the coordinate X^μ or its derivatives.

For the open string, the equations of motion follow from varying the action only if we also require the absence of boundary terms. From:

$$S = -\frac{T}{2}\int_0^\pi d\sigma \int dt\ \partial_a X^\mu \partial^a X_\mu$$

we see that

$$\delta S = T\int_0^\pi d\sigma \int dt\ \delta X^\mu\, \partial_a\, \partial^a X_\mu - T\int dt\left[\delta X^\mu\, \partial_\sigma X_\mu\right]_0^\pi$$

To make the second term vanish, we must require:

$$\delta X^\mu(0, t)\, \partial_\sigma X_\mu(0, t) = \delta X^\mu(\pi, t)\, \partial_\sigma X_\mu(\pi, t) = 0$$

Thus, at $\sigma = 0$, we can impose one of the following two boundary conditions on each of the spacetime coordinates X^μ:

$$\partial_\sigma X^\mu(0,t) = 0 \qquad \text{(Neumann)}$$
$$X^\mu(0,t) = c^\mu \qquad \text{(Dirichlet)}$$

where c^μ is an arbitrary constant.

At the other end $\sigma = l$, we must also independently choose one of these conditions.

Thus an open string can be Neumann-Neumann (NN), Dirichlet-Dirichlet (DD) or Neumann-Dirichlet (ND), with respect to each of its spacetime coordinates.

The D boundary condition violates translation and Lorentz invariance. In fact, it is the statement that the end of the string is stuck at a particular location.

The locus on which the end is stuck is called a Dirichlet brane or D-brane for short.

In the DD case, the two ends can be stuck at the same location c^μ or at two different locations c^μ, d^μ. In one case, the string starts and ends on the same brane, while in the other, it stretches between two different branes.

With open-string boundary conditions, the mode expansion is different from that for the closed string.

A wave travelling one way on the string hits the end, and gets reflected back. So there is only one set of modes rather than two.

For NN boundary conditions, we find:

$$X^\mu(\sigma, t) = x^\mu + 2\alpha' p^\mu t + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-int} \cos n\sigma$$

and

$$\partial_\pm X^\mu(\sigma, t) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-in(t \pm \sigma)}$$

where $\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$.

The Virasoro constraints reduce to the vanishing of a single set of modes:

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{-m} \cdot \alpha_{n+m} = 0$$

For DD boundary conditions the result is:

$$X^\mu(\sigma,t) = c^\mu\left(1-\frac{\sigma}{\pi}\right) + d^\mu\sigma - \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\, e^{-int}\,\sin n\sigma$$

Here c^μ, d^μ specify the locations of the D-branes on which the ends of the string are fixed. As one would expect, there are no translational zero modes x^μ, p^μ in this case.

For DN and ND strings, the mode expansion involves half-integer modes, as one can easily check.