2. Quantising the Bosonic String

(i) Open strings: the role of the constraints

Quantisation of the bosonic string would be extremely simple if it were not for the constraints.

Let us define the canonical momentum conjugate to X^μ by

$$P_\mu(\sigma, t) = \frac{\delta S}{\delta(\partial_t X^\mu(\sigma, t))} = T\,\partial_t X_\mu(\sigma, t)$$

and impose equal-time canonical commutators on P_μ, X^μ:

$$[P_\mu(\sigma, t), X^\nu(\sigma', t)] = -i\delta(\sigma - \sigma')\delta_\mu^\nu$$

$$[X^\mu(\sigma, t), X^\nu(\sigma', t)] = [P^\mu(\sigma, t), P^\nu(\sigma', t)] = 0$$

Now consider for definiteness the open string with NN boundary conditions. The canonical commutators define commutation relations among the modes α_n, x^μ, p^μ:

$$[\alpha_m^\mu, \alpha_n^\nu] = m\, \delta_{m+n,0}\, \eta^{\mu\nu}, \quad [x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_m^\mu, x^\nu] = [\alpha_m^\mu, p^\nu] = 0$$

From the classical relation $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$, it follows that the corresponding operators satisfy

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$$

By analogy with the harmonic oscillator, the operators α_n^μ are creation operators for $n < 0$ and annihilation operators for $n > 0$.

Thus the ground state $|0\rangle$ of the string would be defined by

$$\alpha_n^\mu |0\rangle = 0, \;\; n > 0$$

Physical states would then be constructed by acting with the oscillators α_{-n}^μ on the ground state $|0\rangle$. These would correspond to excited states of the string.

However, we can see right away that something is wrong.

Consider the state $\alpha^{\mu}_{-n}|0\rangle$. Assuming the ground state to be normalised, this excited state would have the norm:

$$\|\alpha^{\mu}_{-n}|0\rangle\|^2 = \langle 0|\alpha^{\mu}_{n}\,\alpha^{\mu}_{-n}|0\rangle = n\,\eta^{\mu\mu}$$

Thus for $\mu = 0$ (the time direction) we have negative-norm states, which are unacceptable in any physical theory.

The constraints play the essential role of eliminating such states.

There are several ways to implement the constraints. Broadly they fall into two categories:

(I) Solve the constraints by singling out some spacetime directions from others. As a result some oscillators become dependent on others and are not to be independently quantised. The Hilbert space then manifestly has positive norm, but Lorentz invariance is not manifest.

(II) Quantise all the oscillators, but then impose the constraints on the Hilbert space. Only those states satisfying the constraints will be treated as physical. Then the full Hilbert space will have negative-norm states, but the constrained subspace will be positive. In this procedure, manifest Lorentz invariance can be maintained.

Instead of choosing one of these two approaches, we will exhibit the flavour of both. Both approaches have important merits.

(ii) Open strings: light-cone gauge

We may try to directly solve the constraint:

$$\partial_- X^\mu \partial_- X_\mu = 0$$

by eliminating, say, X^0 in favour of the other coordinates. However, this will introduce square roots and the equations become very hard to solve.

A more convenient way to solve the constraints is to use the light-cone gauge. Pick two of the D spacetime coordinates μ, for example $\mu = 0$ and $\mu = D-1$, and define:

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$$

If the remaining space coordinates are denoted X^i, $i = 1, 2, \dots, D-2$, then the Virasoro constraint above becomes:

$$-2\,\partial_- X^+ \,\partial_- X^- + \partial_- X^i \,\partial_- X^i = 0$$

In conformal gauge, we still have the freedom to reparametrise the worldsheet coordinates $\xi^{\pm}$ by:

$$\xi^{+} \to \xi'^{+}(\xi^{+}), \qquad \xi^{-} \to \xi'^{-}(\xi^{-})$$

since this preserves the fact that the metric g_{ab} is proportional to η_{ab}.

For such a reparametrisation, $t'(\sigma, t)$ satisfies

$$\partial_{+}\partial_{-}t'(\sigma, t) = 0$$

This is the same equation as the one satisfied by the X^{μ}. So we can use it to choose t proportional to one of the X^{μ}, in particular to X^{+}.

More precisely, we set:

$$X^{+}(\sigma, t) = x^{+} + 2\alpha' p^{+} t$$

where x^{+}, p^{+} are constants. This amounts to eliminating all the oscillators in X^{+}, and keeping only the zero modes.

Using $\partial_- X^+ = \alpha' p^+$, we can solve the constraints for X^-:

$$\partial_- X^- = \frac{1}{2\alpha' p^+} \partial_- X^i \, \partial_- X^i$$

Inserting the mode expansion of X^-, we find that

$$\alpha_n^- = \frac{1}{2\sqrt{2\alpha'}p^+} \sum_{m \in \mathbb{Z}} \alpha_{n-m}^i \, \alpha_m^i$$

This determines all modes of X^- in terms of those of X^i. In particular,

$$p^- = \frac{1}{\sqrt{2\alpha'}} \alpha_0^- = \frac{1}{2\alpha' p^+} \sum_{m=1}^{\infty} \alpha_{-m}^i \, \alpha_m^i + \frac{1}{2p^+} p^i p^i$$

This can be rewritten as

$$2p^+ p^- - p^i p^i = \frac{1}{\alpha'} \sum_{m=1}^{\infty} \alpha_{-m}^i \, \alpha_m^i$$

This leaves the independent oscillators α_n^i, $i = 1, 2, \ldots, D-2$, as well as the zero modes x^i, p^i. These are canonically quantised:

$$[\alpha_m^i, \alpha_n^j] = m\,\delta_{m+n,0}\,\delta^{ij}, \quad [x^i, p^j] = i\delta^{ij}, \quad [\alpha_m^i, x^j] = [\alpha_m^i, p^j] = 0$$

Physical states of the string are now constructed by defining a ground state $|0\rangle$ satisfying

$$\alpha_n^i|0\rangle = 0, \ n > 0$$

We also define the state

$$|k\rangle = e^{ik^i x^i}|0\rangle$$

satisfying

$$p^i|k\rangle = k^i|k\rangle$$

which represents a string in its ground state with transverse momentum k^i.

The excited states of the string will then be of the form:

$$\alpha^{i_1}_{-n_1}\,\alpha^{i_2}_{-n_2}\ldots\alpha^{i_N}_{-n_N}\,|k\rangle$$

These transform as tensors under the transverse rotations, $SO(D-2)$. Full Lorentz symmetry, $SO(D-1,1)$, is not manifest.

What are the masses of these states? We have the standard relativistic formula:

$$M^2 = -p^\mu p_\mu = 2p^+p^- - p^i p^i$$

From a previous equation, we see that:

$$M^2 = \frac{1}{\alpha'}\sum_{m=1}^{\infty}\alpha^i_{-m}\,\alpha^i_m = \frac{1}{\alpha'}\sum_{m=1}^{\infty} m\,N_m$$

where

$$N_m = \frac{1}{m}\alpha^i_{-m}\,\alpha^i_m$$

is the number operator that counts the number of excitations of mode number m.

Thus, we have:

$$M^2|0\rangle = 0, \quad M^2(\alpha^i_{-1}|0\rangle) = \frac{1}{\alpha'}(\alpha^i_{-1}|0\rangle)$$

so apparently the ground state is massless and the first excited state has $(\text{mass})^2 = 1/\alpha'$.

We have just encountered a serious paradox.

The first excited state is massive and has $D - 2$ physical components. In fact, it is a vector of $SO(D - 2)$.

But according to the representation theory of the Lorentz group, only a massless state in D spacetime dimensions can have $D - 2$ physical components. A massive state necessarily has $D - 1$ components!

Thus the first excited state of the string really ought to be massless. If we cannot ensure this, then we have lost/misplaced our Lorentz invariance, and that is the end of string theory...

Fortunately, string theory does manage to evade this paradox.

To see this, we must go back to the equation:

$$M^2 = \frac{1}{\alpha'} \sum_{m=1}^{\infty} \alpha^i_{-m}\, \alpha^i_m = \frac{1}{\alpha'} \sum_{m=1}^{\infty} m\, N_m$$

As a classical formula it is perfectly correct. But after quantising the oscillators α^i_m, the formula develops an ordering ambiguity.

The operators α^i_m and α^i_{-m} do not commute. Indeed,

$$\alpha^i_{-m}\, \alpha^i_m = \alpha^i_m\, \alpha^i_{-m} - m$$

Since we allowed the interchange of orderings at the classical level, we must allow for an additive constant in the formula for M^2.

Thus, the corrected formula is:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} \alpha^i_{-m}\,\alpha^i_m - a\right) = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} m\,N_m - a\right)$$

where a is a constant.

We can determine a by a heuristic argument. Originally, the classical expression arose from the term:

$$\frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha^i_{-m}\,\alpha^i_m$$

If we assume that the oscillators in this expression are replaced by quantum operators in the same ordering, then we must re-order precisely half of the terms to get the normal-ordered expression:

$$\frac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha^i_{-m}\,\alpha^i_m: \;=\; \sum_{m=1}^{\infty} \alpha^i_{-m}\,\alpha^i_m$$

It is easy to see that

$$\frac{1}{2}\sum_{m=-\infty}^{\infty}\alpha_{-m}^{i}\,\alpha_{m}^{i}=\frac{1}{2}\sum_{m=-\infty}^{\infty}:\alpha_{-m}^{i}\,\alpha_{m}^{i}:\;+\frac{D-2}{2}\sum_{m=1}^{\infty}m$$

With zeta-function regularisation,

$$\sum_{m=1}^{\infty}m=-\frac{1}{12}$$

so, if we believe this, then we have determined

$$a=\frac{D-2}{24}$$

But we have already determined a by consistency with the representation theory of the Lorentz group. It must be such as to render the first excited state massless. This happens only if $a = 1$.

It follows that

$$D = 26$$

so the bosonic string is consistent only in 26 spacetime dimensions!

There is another interesting consequence. The mass formula has become:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} m\, N_m - 1\right)$$

This implies that the ground state is tachyonic:

$$M^2|0\rangle = -\frac{1}{\alpha'}|0\rangle$$

Have we solved one problem only to encounter another, worse one?

No. Once we accept 26 dimensions and a tachyon, everything else actually falls into place very nicely.

Indeed, one can show that the degeneracies of all the excited states

$$\alpha^{i_1}_{-n_1}\,\alpha^{i_2}_{-n_2}\ldots\alpha^{i_N}_{-n_N}\,|k\rangle$$

match the dimensions of irreps of the little group $SO(D-1)$.

For example, at the second excited level we have:

$$\alpha^i_{-2}|0\rangle,\quad \alpha^i_{-1}\alpha^j_{-1}|0\rangle$$

Each of these separately cannot be a representation of $SO(D-1)$. But together, they can. The total number of states is

$$(D-2)+\frac{1}{2}(D-2)(D-1)=\frac{1}{2}(D-2)(D+1)$$

which is precisely the dimension of the symmetric, traceless representation of the little group $SO(D-1)$.

(iii) Open strings: covariant quantisation and gauge invariance

We have seen that the NN open bosonic string has a tachyonic ground state and a massless vector as its first excited state.

At low energies, one may expect that these are described by a scalar field $\phi(x)$ and a vector field $A_\mu(x)$, with a field-theory Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2, \qquad m^2 = -1/\alpha'$$

The gauge field action has the well-known Abelian gauge invariance:

$$\delta A_\mu = \partial_\mu \Lambda(x)$$

If the free action was not gauge invariant, the field theory would have negative-norm states, as we learn in electrodynamics.

Can we find direct evidence for gauge invariance from string theory? This would be strong confirmation that at low energies, string theory is described by familiar field theories.

In light-cone gauge, we worked only with physical degrees of freedom, so we could not have seen gauge invariance.

Instead, we now briefly consider covariant quantisation of the string. In this formalism, we quantise all the oscillators.

Consider a general linear combination of the first excited states:

$$\zeta_\mu(k)\,\alpha^\mu_{-1}|k\rangle$$

It is natural to identify $\zeta_\mu(k)$ with the Fourier transform of the gauge field $A_\mu(x)$.

Now we impose the constraints:

$$L_n|\mathsf{phys}\rangle = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{n+m}|\mathsf{phys}\rangle = 0$$

to define states $|\mathsf{phys}\rangle$ that are in the physical Hilbert space.

It is easily verified that L_n, L_{-n} do not commute with each other in general. Their commutator includes a c-number term.

This means we cannot require both

$$L_n|\text{phys}\rangle = 0 \;\; \text{and} \;\; L_{-n}|\text{phys}\rangle = 0$$

so in covariant quantisation we only impose

$$L_n|\text{phys}\rangle = 0, \;\; n \geq 0$$

Applying this requirement on

$$|\text{phys}\rangle = \zeta_\mu(k)\,\alpha^\mu_{-1}|k\rangle$$

we find that only L_0, L_1 give nontrivial constraints, which turn out to be:

$$k_\mu k^\mu\,\zeta_\nu(k) = 0, \qquad k^\mu\,\zeta_\mu(k) = 0$$

The first condition says the field is massless, which we know. Taken together, the two conditions give us the free-field equation of motion:

$$k^\mu\,(k_\mu\zeta_\nu - k_\nu\zeta_\mu) = 0 \quad \leftrightarrow \quad \partial^\mu F_{\mu\nu} = 0$$

Now consider the state

$$L_{-1}|k\rangle \sim k_\mu\, \alpha^\mu_{-1}|k\rangle$$

This is clearly a massless state, but it is unphysical in the following sense. We have

$$\langle \text{phys}|L_{-1}|k\rangle = 0$$

for every physical state $\langle \text{phys}|$, because

$$L_1|\text{phys}\rangle = 0 \;\Rightarrow\; \langle \text{phys}|L_{-1} = 0$$

So the state is orthogonal to the physical Hilbert space, and is equivalent to zero. Thus for arbitrary $\Lambda(k)$, we have the equivalence of polarisation vectors:

$$\zeta_\mu(k) \sim \zeta_\mu(k) + k_\mu\, \Lambda(k)$$

This is just the momentum space version of the gauge equivalence:

$$A_\mu \sim A_\mu + \partial_\mu \Lambda$$

Thus, at least at the non-interacting level, string theory has gauge invariance. We did not require it, rather it emerged upon quantising the theory.

This is impressive confirmation that string theory has a profound degree of internal consistency. It embodies symmetry principles that are fundamental to particle physics.

To summarise, the open bosonic string has a tachyon and a massless photon in its spectrum.

There is also an infinite tower of higher excited states, whose mass^2 are integrally spaced in units of $1/\alpha'$.

(iv) Closed strings: spectrum and gauge invariance

Without much effort, we can repeat the same steps for the closed bosonic string. As we have seen, there are twice as many oscillators, because of the independent left- and right-moving sectors.

However, the zero modes x^μ, p^μ are not doubled.

Both $L_0, \widetilde{L}_0$ have normal ordering ambiguities and must have a constant $-1/\alpha'$ subtracted from them. We find:

$$M^2 = L_0 + \widetilde{L}_0 = \frac{2}{\alpha'}\left(\sum_{m=1}^{\infty} \alpha^i_{-m}\,\alpha^i_m + \sum_{m=1}^{\infty} \widetilde{\alpha}^i_{-m}\,\widetilde{\alpha}^i_m - 2\right)$$

It follows that the ground state $|k\rangle$ is a tachyon of

$$M^2 = -\frac{4}{\alpha'}$$

What about excited states?

In principle we could act with any number of oscillators $\alpha_{-n}^i, \tilde{\alpha}_{-n}^i$ on the ground state.

But there is a subtlety. The zero mode p^- is common for both left- and right-movers. So the procedure we followed for the open string now gives:

$$p^- = \frac{1}{\alpha' p^+}\left(\sum_{m=1}^{\infty} \alpha_{-m}^i\, \alpha_m^i - 1\right) + \frac{1}{2p^+} p^i p^i$$

$$p^- = \frac{1}{\alpha' p^+}\left(\sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i\, \tilde{\alpha}_m^i - 1\right) + \frac{1}{2p^+} p^i p^i$$

and hence:

$$\sum_{m=1}^{\infty} \alpha_{-m}^i\, \alpha_m^i = \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i\, \tilde{\alpha}_m^i$$

Thus the total level $\sum nN_n$ must be equal for left- and right-movers. This rules out the states

$$\alpha^i_{-1}|k\rangle, \quad \tilde{\alpha}^i_{-1}|k\rangle$$

and therefore the first excited state of the closed bosonic string is:

$$\alpha^i_{-1}\,\tilde{\alpha}^j_{-1}|k\rangle$$

which is massless. A general linear combination is

$$\zeta_{ij}(k)\,\alpha^i_{-1}\,\tilde{\alpha}^j_{-1}|k\rangle$$

As a representation of $SO(D-2)$, $\zeta_{ij}(k)$ decomposes into three irreducible representations: symmetric traceless, antisymmetric, and a trace part which is a singlet.

Each one can be identified with the transverse components of a field:

$$
\begin{aligned}
\zeta_{(ij)}(k) - \tfrac{1}{D-2}\,\delta^{ij}\,\zeta_{ij}(k) &\rightarrow G_{ij}(x) \\
\zeta_{[ij]}(k) &\rightarrow B_{ij}(x) \\
\delta^{ij}\,\zeta_{ij}(k) &\rightarrow \Phi(x)
\end{aligned}
$$

These fields, in turn, are the transverse components of the massless fields $G_{\mu\nu}, B_{\mu\nu}, \Phi$ of the Lorentz group $SO(25, 1)$.

Thus we have shown that the massless first excited state of the bosonic string consists of these three fields. They should be described at low energies by a suitable field theory action.

But it is a theorem that the only consistent action for a massless symmetric rank-2 tensor field is that of Einstein's gravity.

Therefore, closed string theory, if consistent, is a theory of gravity!.

We therefore expect the low-energy effective action to be:

$$\mathcal{L} = \sqrt{-\|G\|}\Big(-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + R$$
$$- \partial_{[\mu}B_{\nu\lambda]}\,\partial^{[\mu}B^{\nu\lambda]} - \frac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi\Big),\;\; m^2 = -4/\alpha'$$

which is to be taken seriously only to quadratic order in the fields. For the metric this means we make the linearised approximation:

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and keep terms quadratic in $h_{\mu\nu}$.

If this is true, the symmetries that should be visible (at the level of inhomogeneous transformations) are:

(I) linearised reparametrisation invariance:

$$\delta h_{\mu\nu}(x) = \partial_\mu \Lambda_\nu(x) + \partial_\nu \Lambda_\mu(x)$$

(II) tensor gauge invariance:

$$\delta B_{\mu\nu}(x) = \partial_\mu \Lambda_\nu(x) - \partial_\nu \Lambda_\mu(x)$$

In each case, $\Lambda_\mu(x)$ is a vector gauge parameter.

In the first symmetry transformation, $\Lambda_\mu(x)$ labels an infinitesimal reparametrisation in spacetime.

The second transformation is less familiar, but it is known to remove negative-norm states in the field theory of a second-rank antisymmetric tensor field (which we call a 2-form field for short).

By now we know how to check this.

Carry out covariant quantisation of the closed string, and require that the constraints annihilate the state:

$$\zeta_{\mu\nu}(k)\,\alpha^{\mu}_{-1}\,\widetilde{\alpha}^{\nu}_{-1}|k\rangle$$

This leads to the linearised equations of motion .

Next, allow addition of the unphysical states:

$$\Lambda_{\mu}(k)\,\left(L_{-1}\widetilde{\alpha}^{\mu}_{-1}\pm\alpha^{\mu}_{-1}\widetilde{L}_{-1}\right)|k\rangle$$

and obtain the gauge equivalences:

$$\zeta_{\mu\nu}\sim\zeta_{\mu\nu}+\left(k_{\mu}\,\Lambda_{\nu}\pm k_{\nu}\,\Lambda_{\mu}\right)$$

as desired.

The trace part of the state obtained in light-cone quantisation corresponds to a scalar field $\Phi(x)$ called the dilaton. This field plays a very important role in string theory, and we will have more to say about it later.

To summarise, the closed bosonic string has a tachyon and a massless graviton, a 2-form field and a dilaton.

There is also an infinite tower of higher excited states whose mass^2 are integrally spaced in units of $4/\alpha'$.

Between open and closed strings, the twin principles of gauge invariance and gravity are embodied. The price we have paid is 26 dimensions and a tachyon.