We would like to have worldsheet supersymmetry between X^μ and ψ^μ_α. But since there is also an auxiliary variable g_{ab} on the worldsheet, we introduce its fermionic superpartner $\chi_{\alpha a}$, rather like a worldsheet gravitino.

The action will have local worldsheet supersymmetry as well as worldsheet reparametrisation invariance. There is no kinetic term for the graviton and gravitino.

Thus, the action is that of non-dynamical worldsheet supergravity coupled to supersymmetric matter:

$$S = -\frac{T}{2}\int d\sigma\, dt\, \sqrt{-\|g\|}\Big(g^{ab}\partial_a X^\mu \partial_b X_\mu - i\overline{\psi}^\mu \rho^a \partial_a \psi_\mu \\ + 2\,\overline{\chi}_a\, \rho^b \rho^a \psi^\mu \partial_b X_\mu + \frac{1}{2}\,\overline{\psi}_\mu \psi^\mu\, \overline{\chi}_a\, \rho^b \rho^a \chi_b\Big)$$

Here, ρ^a are 2×2 gamma-matrices, and all spinor indices have been suppressed.

As promised, this action has local worldsheet supersymmetry as well as reparametrisation invariance. As before, we can use reparametrisations (and Weyl invariance) to effectively fix the metric $g_{ab} = \eta_{ab}$.

Less obvious, but equally true, is that we can use local supersymmetry as well as a super version of Weyl invariance, to effectively set $\chi_{\alpha a} = 0$.

The result is an action in superconformal gauge, which is rather simple:

$$S = -\frac{T}{2} \int d\sigma \, dt \left(\partial_a X^\mu \partial_a X_\mu - i \overline{\psi}^\mu \rho^a \partial_a \psi_\mu \right)$$

The equations of motion of the bosonic and fermionic coordinates are the familiar Klein-Gordon and Dirac equations in two dimensions:

$$\partial_a \partial^a X^\mu = 0, \qquad \rho^a \partial_a \psi^\mu = 0$$

In light-cone coordinates, and with a decomposition of ψ^μ_α into its chiral components $\psi_\pm$ in a suitable basis of gamma-matrices, these become:

$$\partial_-\partial_+X^\mu = 0, \qquad \partial_-\psi^\mu_+ = 0, \quad \partial_+\psi^\mu_- = 0$$

The superconformal gauge action has global worldsheet supersymmetry:

$$\delta X^\mu = \bar\epsilon\psi^\mu, \qquad \delta\psi^\mu = -i\rho^a\partial_a X^\mu$$

Correspondingly, there is a conserved supercurrent:

$$J_{\alpha a} = \frac{1}{2}\,\rho^b\rho_a\,\psi^\mu_\alpha\,\partial_b X_\mu, \qquad \partial^a J_{\alpha a} = 0$$

The gravitino equations of motion, derived from the original action, reduce in superconformal gauge to the new constraint $J_{\alpha a} = 0$.

Thus the constraints that must be imposed are the vanishing of the energy-momentum tensor T_{ab} and the supercurrent $J_{\alpha a}$.

In superconformal gauge and light-cone coordinates, these "Super-Virasoro" constraints are:

$$T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu\, \partial_- \psi_{-\,\mu} = 0$$

$$J_{--} = \psi_-^\mu\, \partial_- X_\mu = 0$$

along with their left-moving $(+)$ counterparts.

Next we would like to make a mode expansion of the fermionic coordinates ψ^μ. As we are doing closed strings, they would naively be periodic in σ, but it is equally consistent to take them antiperiodic in σ.

For historical reasons, we assign the names:

$$\begin{aligned} \text{periodic:} \quad & \psi^\mu(\sigma+\pi,t) = \psi^\mu(\sigma,t) \quad \longrightarrow \ \text{Ramond (R)} \\ \text{antiperiodic:} \quad & \psi^\mu(\sigma+\pi,t) = -\psi^\mu(\sigma,t) \ \longrightarrow \ \text{Neveu-Schwarz (NS)} \end{aligned}$$

and we will see that both are important.

The mode expansions are then:

$$\text{R sector}: \psi_-^\mu(\sigma,t) = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} d_n^\mu \, e^{-2in(t-\sigma)}$$

$$\text{NS sector}: \psi_-^\mu(\sigma,t) = \sqrt{2\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu \, e^{-2ir(t-\sigma)}$$

along with their left-moving counterparts.

Now we can write the constraints in terms of bosonic and fermionic modes:

R sector:

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{n+m} + \frac{1}{2}\sum_{m\in\mathbb{Z}}\left(\frac{n}{2}+m\right) d_{-m}\cdot d_{n+m} = 0$$

$$G_n = \sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot d_{m+n} = 0$$

NS sector:

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{n+m} + \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(\frac{n}{2}+r\right) b_{-r}\cdot b_{n+r} = 0$$

$$G_s = \sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot b_{m+s} = 0$$

with similar expressions for the left-movers.

Note that at this stage the NS/R boundary conditions can be chosen independently for right and left movers.

(ii) The open fermionic string

For the open string, the boundary conditions in the variation of the fermionic part of the action are easily seen to be:

$$\int dt \left[\delta\psi_+^\mu\,\psi_{+\mu} - \delta\psi_-^\mu\,\psi_{-\mu}\right]_0^\pi = 0$$

The solution of these conditions is:

$$\psi_-^\mu(0,t) = \eta_1\,\psi_+^\mu(0,t)$$
$$\psi_-^\mu(\pi,t) = \eta_2\,\psi_+^\mu(\pi,t)$$

where $\eta_1, \eta_2 = \pm 1$.

While the sign can be chosen independently at each end, the terminology depends on the relative sign:

$$\eta_1 = \eta_2 : \ \text{Ramond}, \qquad \eta_1 = -\eta_2 : \ \text{Neveu-Schwarz}$$

The mode expansion for the worldsheet fermion of an open string is in terms of integer modes d_n in the R sector and half-integer modes b_r in the NS sector.

As long as we consider NN or DD open strings, the Super-Virasoro constraints for the open string are the same as for one sector (left- or right-movers) of the closed string.

For DN, ND strings, however, the bosonic oscillators also become half-integer moded and we must modify the sums in the constraints accordingly.