4. Quantising the Superstring

(i) Open string: Neveu-Schwarz sector

Quantisation of the fermionic string proceeds along very similar lines as the bosonic string. So we can be less explicit about the familiar details.

But there will be some new subtleties to take care of, mainly having to do with boundary conditions on the fermions.

Let us perform light-cone quantisation of the action:

$$S = -2T\int d\sigma\, dt\,\Big(\partial_+ X^\mu \partial_- X_\mu + i\psi_+^\mu \partial_- \psi_{+\mu} + i\psi_-^\mu \partial_+ \psi_{-\mu}\Big)$$

together with the constraints:

$$T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu\, \partial_- \psi_{-\,\mu} = 0$$

$$J_{--} = \psi_-^\mu\, \partial_- X_\mu = 0$$

As before, we make the choice:

$$X^+ = x^+ + 2\alpha' p^+ t$$

Recall that this choice is made using the local reparametrisations that preserve conformal gauge, which satisfy

$$\partial_+ \partial_- t'(\sigma, t) = 0$$

It seems natural that we should fix ψ^+ to be similarly free of oscillators. For this, we need to consider the local supersymmetry transformations that preserve superconformal gauge.

The analysis, which we will skip, reveals that these are given by supersymmetry parameters $\epsilon_\pm$ satisfying

$$\partial_+\epsilon_- = \partial_-\epsilon_+ = 0$$

Again these are the same equations as those satisfied by $\psi^\mu_\pm$. We can therefore set

$$\psi^+_\pm = 0$$

Imposing these gauge choices, we find that

$$\begin{aligned}\partial_- X^- &= \frac{1}{2\alpha' p^+}\left(\partial_- X^i\,\partial_- X^i + + \frac{i}{2}\psi_-\partial_-\psi_-\right)\\ \psi^-_- &= \frac{1}{2\alpha' p^+}\psi^i_-\partial_- X^i\end{aligned}$$

Let us now choose Neveu-Schwarz boundary conditions. Recall that in this sector, the open string satisfies:

$$\psi_-^\mu(0,t) = \psi_+^\mu(0,t)$$
$$\psi_-^\mu(\pi,t) = -\psi_+^\mu(\pi,t)$$

Therefore it has a half-integer mode expansion:

$$\psi_-^\mu(\sigma,t) = \sqrt{\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu \, e^{-ir(t-\sigma)}$$
$$\psi_+^\mu(\sigma,t) = \sqrt{\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu \, e^{-ir(t+\sigma)}$$

Note the absence of any zero-frequency mode.

In terms of modes, the light-cone gauge condition leads to:

$$\alpha_n^- = \frac{1}{2\sqrt{2\alpha' p^+}}\left(\sum_{m\in\mathbb{Z}} :\alpha^i_{n-m}\,\alpha^i_m: + \sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(r - \frac{n}{2}\right) :b^i_{n-r}b^i_r: - a\,\delta_{n,0}\right)$$

$$b_r^- = \frac{1}{\sqrt{2\alpha' p^+}}\sum_{s\in\mathbb{Z}+\frac{1}{2}} \alpha^i_{r-s} b^i_s$$

As before, we have included a possible normal-ordering constant a in α_0^-.

Canonical quantisation leads to the anticommutators:

$$\{b^i_r,\, b^j_s\} = \delta_{r+s,0}\,\delta^{ij}$$

What is the normal-ordering constant this time? If we calculate it as before by evaluating a divergent sum, we find:

$$a = \frac{D-2}{16}$$

Now consider the ground state $|k\rangle$ annihilated by all positive modes:

$$\alpha_n^i|k\rangle = b_r^i|k\rangle = 0, \quad n > 0,\ r > 0$$

At the first excited level, we get a vector state:

$$b^i_{-\frac{1}{2}}|k\rangle$$

This state again has only $D-2$ components and therefore has to be massless. This fixes

$$a = \frac{1}{2}$$

and therefore also

$$D = 10$$

Thus we see that the fermionic string is consistent in 10 spacetime dimensions.

The mass-shell condition is now:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} \alpha_{-m}^i\, \alpha_m^i + \sum_{r=\frac{1}{2}}^{\infty} r\, b_{-r}^i\, b_r^i - \frac{1}{2}\right)$$

So this sector appears to have a tachyon, with

$$M^2 = -\frac{1}{2\alpha'}$$

We will soon demonstrate that this is not quite true. First we have to consider the Ramond sector.

(ii) Open string: Ramond sector

In this sector, the fermions have integer modes:

$$\psi_-^\mu(\sigma,t) = \sqrt{\alpha'} \sum_{n\in\mathbb{Z}} d_n^\mu \, e^{-in(t-\sigma)}$$

$$\psi_+^\mu(\sigma,t) = \sqrt{\alpha'} \sum_{n\in\mathbb{Z}} d_n^\mu \, e^{-in(t+\sigma)}$$

and the canonical anticommutators are:

$$\{d_m^i,\, d_n^j\} = \delta_{m+n,0}\,\delta^{ij}$$

In this sector, it turns out that the normal-ordering constant a is zero. This is a consequence of worldsheet supersymmetry (which was broken by boundary conditions in the NS sector).

Thus the mass-shell condition is:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} \alpha_{-m}^i\,\alpha_m^i + \sum_{n=1}^{\infty} n\, d_{-n}^i\, d_n^i\right)$$

Because of the zero-frequency mode d_0^i, the quantisation of this sector is somewhat novel.

The ground state is defined, as usual, by

$$\alpha_n^i|k\rangle = d_n^i|k\rangle = 0, \quad n > 0$$

This state clearly has $M^2 = 0$.

The operators $\alpha_{-n}^i, d_{-n}^i, \; n > 0$ are creation operators for the excited states:

$$\alpha_{-n_1}^{i_1}\,\alpha_{-n_2}^{i_2}\ldots\alpha_{-n_N}^{i_N}\,d_{-r_1}^{j_1}\,d_{-r_2}^{j_2}\ldots d_{-r_M}^{j_M}\,|k\rangle$$

But what about the zero modes d_0^i ? They satisfy the Clifford algebra:

$$\{d_0^i,\, d_0^j\} = \delta^{ij}$$

which in particular tells us that

$$\left(d_0^i\right)^2 = 1, \quad \text{all } i$$

Now consider the state $d_0^i|k\rangle$. Because of the above relation, this state cannot be zero. On the other hand, since

$$[M^2, d_0^i] = 0$$

we see that $d_0^i|k\rangle$ has the same value of M^2 as $|k\rangle$, namely $M^2 = 0$.

Thus we have the phenomenon of ground-state degeneracy.

Given a ground state in the R sector, we can generate a degenerate set of ground states:

$$|k\rangle, \quad d_0^i|k\rangle, \quad d_0^i\, d_0^j|k\rangle, \cdots$$

The smallest such set is the irreducible representation of the Clifford algebra, obtained by taking the combinations:

$$D^1 = d_0^1 + i d_0^2, \quad D^2 = d_0^3 + i d_0^4, \quad D^3 = d_0^5 + i d_0^6, \quad D^4 = d_0^7 + i d_0^8$$

which obey

$$\{D^I, D^{J\dagger}\} = 2\delta^{IJ}, \qquad \{D^I, D^J\} = 0$$

Then we can consistently require $|k\rangle$ to satisfy:

$$D^I|k\rangle = 0, \quad I = 1,2,3,4$$

and the only nonzero states are

$$|k\rangle, \quad D^{I\dagger}|k\rangle, \quad D^{I\dagger}D^{J\dagger}|k\rangle, \quad D^{I\dagger}D^{J\dagger}D^{K\dagger}|k\rangle, \quad D^{I\dagger}D^{J\dagger}D^{K\dagger}D^{L\dagger}|k\rangle$$

Because of antisymmetry, there are $2^4 = 16$ such states.

This is understandable, since the d_0^i satisfy the anticommutation relations of gamma-matrices in 8 Euclidean dimensions, whose dimension is $2^{\frac{8}{2}} = 16$. This implies that the 16 ground states transform as the spinor representation of $SO(8)$ (and, less obviously, of $SO(9,1)$).

By the spin-statistics theorem, these must therefore be fermions in spacetime!

These 16-component spinors are reducible into two 8-component spinors, one of each chirality. Thus the Ramond sector of the open fermionic string has two massless spacetime fermions, one of each chirality.

We label these as s (spinor) and c (conjugate spinor):

$$|\alpha, k\rangle_s, \quad |\alpha, k\rangle_c, \quad \alpha = 1, 2, \ldots 8$$

The excited states in this sector are obtained by acting with α^i_{-n}, d^i_{-n} on this spinorial ground state. That means they are all spacetime fermions.

(iii) Open string: The GSO projection

The open string spectrum that we have found in the NS and R sectors will be displayed, for the lowest levels, in a table on the following page.

In the NS sector there are states at every half-integer level, while the R sector only has states at integer levels.

We will see an interesting pattern. At the integer levels, the degeneracy of the R states is exactly double that of the NS states.

Note also that all the NS states are spacetime bosons, while all the R states are spacetime fermions.

$\alpha' M^2$	Neveu-Schwarz	Degen.	Ramond	Degen.
$-\frac{1}{2}$	$\vert k\rangle$	1	—	—
0	$b^i_{-\frac{1}{2}}\vert k\rangle$	8	$\vert\alpha; k\rangle_{s,c}$	16
$\frac{1}{2}$	$b^i_{-\frac{1}{2}}\, b^j_{-\frac{1}{2}}\vert k\rangle$	28	—	—
	$\alpha^i_{-1}\vert k\rangle$	8		
1	$b^i_{-\frac{1}{2}}\, b^j_{-\frac{1}{2}}\, b^k_{-\frac{1}{2}}\vert k\rangle$	56	$d^i_{-1}\vert\alpha; k\rangle_{s,c}$	128
	$b^j_{-\frac{3}{2}}\vert k\rangle,\ \alpha^i_{-1} b^j_{-\frac{1}{2}}\vert k\rangle$	8+64	$\alpha^i_{-1}\vert\alpha; k\rangle_{s,c}$	128

We see that the NS sector by itself is unsatisfactory for a physical theory, because it has only bosons. Moreover, it has a tachyon.

The R sector by itself is also unsatisfactory for a physical theory. It has only fermions (though fortunately no tachyon). Moreover, the fermions occur in both chiralities, which makes it harder to obtain a parity violating theory.

If we could somehow combine the two sectors, and also project out the NS states at half-integer levels as well as half the R states at the integer levels, we would have solved all these problems.

This is exactly what we will now do! And the resulting theory will have supersymmetry in spacetime. It will be the open superstring in 10 dimensions.

Projecting out some levels of a string theory is not generally consistent. It can be done consistently only if we take the quotient of the theory by a symmetry operator.

This we consider a simple example of an orbifold. In this case, the symmetry is not a spacetime or geometrical symmetry.

In the NS sector, define F to be the worldsheet fermion number. This is an operator that should satisfy:

$$[F, b^i_r] = b^i_r, \; r < 0$$
$$[F, b^i_r] = -b^i_r, \; r > 0$$

It is easy to see that

$$F = \sum_{r=\frac{1}{2}}^{\infty} b^i_{-r}\, b^i_r$$

Consider the symmetry operator $(-1)^F$. One can check that

$$\{(-1)^F, b_r^i\} = 0, \text{ all } r$$

$$\left((-1)^F\right)^2 = 1$$

In particular the eigenvalues of $(-1)^F$ are ± 1.

We now take the orbifold of the entire NS sector by $(-1)^F$. This means we project in all the states with eigenvalue $+1$, and project out those with eigenvalue -1.

It is up to us to assign the eigenvalue of the NS ground state $|k\rangle$. We choose

$$(-1)^F|k\rangle = -|k\rangle$$

which gets rid of the tachyon.

Moreover, all half-integer levels get projected out, as desired.

In the R sector, the corresponding operator is:

$$(-1)^F = \hat{\Gamma}\ (-1)^{\sum_1^\infty d^i_{-n} d^i_n}$$

The exponential part causes this to anticommute with the d^i_n, while the factor

$$\hat{\Gamma} = \Gamma^1 \Gamma^2 \cdots \Gamma^8$$

anticommutes with the $d^i_0 \sim \Gamma^i$.

On the Ramond ground state, the exponent does not contribute, but $\hat{\Gamma}$, the chirality matrix, projects out one of the two 8-component spinors. Thus at the end we are left with a single massless Majorana-Weyl spinor in 10 spacetime dimensions:

$$|\alpha, k\rangle_s \quad \text{or} \quad |\alpha, k\rangle_c$$

The procedure we have carried out is called the GSO projection. It may look a little artificial, but has a sound basis – as further study (not in this course!) will reveal.

Combining the projected NS and R sectors, we get a theory whose massless sector is a 10-dimensional photon A_μ and a 10-dimensional Majorana-Weyl (real, chiral) fermion λ_α. Both have 8 physical degrees of freedom.

Indeed, at the noninteracting level, they can be combined into a 10 dimensional gauge supermultiplet.

What is much more remarkable is that at every excited level, the bosonic states from the NS sector and the fermionic states from the R sector combine to form massive supermultiplets.

If this procedure of combining and projecting sectors is to be more than an artifice, the interactions should also respect supersymmetry. This is a very restrictive requirement! We will see that it is automatically met by string theory.

(iv) Closed string: The GSO projection

In the closed superstring, we have a very similar structure to that derived above, but independently in the left- and right-moving sectors. Thus, we first carry out the GSO projection in each sector.

The full spectrum is obtained by combining left-and right-movers subject to the constraint that the total mode number of left- and right-moving oscillators is equal.

Thus altogether we have four sectors:

NS-NS, R-R, NS-R, R-NS

In each sector we will have massless states, as well as a tower of massive states spaced in integral multiples of $\frac{4}{\alpha'}$.

In the NS-NS sector we find the massless states:

$$b^i_{-\frac{1}{2}} \widetilde{b}^j_{-\frac{1}{2}} |k\rangle$$

which, just as for the bosonic string, breaks up into a symmetric traceless, antisymmetric and trace part.

In covariant language these are represented by massless fields propagating in 10 spacetime dimensions:

$$G_{\mu\nu}(x), B_{\mu\nu}(x), \Phi(x)$$

In the R-R sector, we have to combine

$$\Big(|\alpha,k\rangle_s \quad \text{or} \quad |\alpha,k\rangle_c\Big)_L \times \Big(|\widetilde{\alpha},k\rangle_s \quad \text{or} \quad |\widetilde{\alpha},k\rangle_c\Big)_R$$

The choice of spinor chirality (s or c) in each sector is a pure convention. But the relative choice between the left- and right-movers is significant.

Thus we have two physically inequivalent choices:

$$\Big(|\alpha,k\rangle_s\Big)_L \times \Big(|\widetilde{\alpha},k\rangle_c\Big)_R \quad \text{or} \quad \Big(|\alpha,k\rangle_s\Big)_L \times \Big(|\widetilde{\alpha},k\rangle_s\Big)_R$$

The product of two spinorial representations of the Lorentz group is a tensorial representation. Thus in both cases, the R-R sector contains only bosons.

Introduce the notation:

$$C^{(r)}_{\mu_1,\mu_2,\ldots,\mu_r}$$

for an rth rank totally antisymmetric tensor field.

A bit of group theory tells us that

$$\Big(|\alpha, k\rangle_s\Big)_L \times \Big(|\tilde{\alpha}, k\rangle_c\Big)_R \;\rightarrow\; C^{(1)}_{\mu}(x),\; C^{(3)}_{\mu\nu\lambda}(x)$$

while

$$\Big(|\alpha, k\rangle_s\Big)_L \times \Big(|\tilde{\alpha}, k\rangle_s\Big)_R \;\rightarrow\; C^{(0)}(x),\; C^{(2)}_{\mu\nu}(x),\; C^{(4)}_{\mu\nu\lambda\rho}(x)$$

These are inequivalent sets of bosonic fields in 10 dimensions. Thus there are two types of theories:

(i) opposite chiralities for left-and right-movers: type IIA

(ii) same chirality for left-and right-movers: type IIB

Finally we look at the NS-R and R-NS sectors. In each case, we are combining a tensor and spinor representation, so the result is spinorial and therefore fermionic. The massless states are as follows:

Type IIA:

$$\text{NS-R}: \quad b_{-\frac{1}{2}}|\widetilde{\alpha}, k\rangle_c$$

$$\text{R-NS}: \quad \widetilde{b}_{-\frac{1}{2}}|\alpha, k\rangle_s$$

Type IIB:

$$\text{NS-R}: \quad b_{-\frac{1}{2}}|\widetilde{\alpha}, k\rangle_s$$

$$\text{R-NS}: \quad \widetilde{b}_{-\frac{1}{2}}|\alpha, k\rangle_s$$

In every case, we get a product of a vector and a spinor. The result includes a Rarita-Schwinger fermion (in 4 dimensions it would have "spin $\frac{3}{2}$"). This is a gravitino, the supersymmetric partner of the graviton.

It has long been known that there are precisely two types of massless supermultiplets in 10 dimensions. These are the building blocks of two classical field theories called type IIA and type IIB supergravity.

Each one has $\mathcal{N}=2$ local supersymmetry in spacetime, and therefore two gravitinos. The two theories are distinguished by the relative chirality of the two gravitinos:

$$\text{opposite} \;\longrightarrow\; \text{IIA}, \qquad \text{same} \;\longrightarrow\; \text{IIB}$$

The field contents are as follows:

$$\begin{array}{lll}
\text{IIA} & \text{bosons :} & G_{\mu\nu},\ B_{\mu\nu},\ \Phi,\ C^{(1)}_{\mu},\ C^{(3)}_{\mu\nu\lambda} \\
& \text{fermions :} & \chi^{\text{left}}_{\mu,\alpha},\ \chi^{\text{right}}_{\mu,\alpha},\ \lambda^{\text{left}}_{\alpha},\ \lambda^{\text{right}}_{\alpha} \\
\text{IIB} & \text{bosons :} & G_{\mu\nu},\ B_{\mu\nu},\ \Phi,\ C^{(0)},\ C^{(2)}_{\mu\nu},\ C^{(4)}_{\mu\nu\lambda\rho} \\
& \text{fermions :} & \chi^{\text{left}}_{\mu,\alpha},\ \chi^{\text{left}}_{\mu,\alpha},\ \lambda^{\text{right}}_{\alpha},\ \lambda^{\text{right}}_{\alpha}
\end{array}$$

The massless spectra of the two closed string theories we have studied are in perfect correspondence with those of the two supergravities.

This suggests that the two types of GSO projections lead to two distinct spacetime supersymmetric string theories, type IIA and type IIB, which are related in some way to the corresponding supergravities.

These two superstring theories, and their cousins, are central to the goal of describing the real world through string theory.