5. Effective actions, Symmetries and Interactions

(i) The effective action

We have suggested that the type IIA and IIB superstring theories are somehow related to the corresponding supergravity theories.

Let us now make the relationship more precise. Supergravity is not a renormalisable quantum field theory. So its action is just a classical object. However, it can very well be the low energy effective action of some well-defined quantum theory.

It is believed that string theory is a well-defined quantum theory. It contains both massless and massive states. On integrating out the massive states, we get an effective action for the massless states, with the same symmetries as the original string theory.

We propose that this effective action is the supergravity action.

Accordingly, let us examine the symmetries of superstring theory at the level of massless states. As with the bosonic string, the covariant formalism must be used here.

We skip the details of the calculation. The results are:

(i) Local reparametrisation invariance:

$$\delta h_{\mu\nu} = \partial_\mu \Lambda_\nu(x) + \partial_\nu \Lambda_\mu(x)$$

(ii) Local supersymmetry:

$$\delta e_\mu^a = \bar{\epsilon}(x)\,\Gamma^a \chi_\mu, \ \cdots$$

(iii) Local p-form gauge invariance:

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu(x) - \partial_\nu \Lambda_\mu(x)$$

$$\delta C^{(p)}_{\mu_1\mu_2\cdots\mu_p} = \partial_{[\mu_1} \Lambda_{\mu_2\cdots\mu_p]}(x)$$

These are all known to be local symmetries of the corresponding linearised supergravity!

In supergravity, these can be extended to symmetries of the interacting action too. It is reasonable to expect that superstring theory likewise incorporates these symmetries even when we introduce interactions.

Before asking how superstring interactions are computed, let us take a schematic look at a part of the supergravity action (avoiding numerical constants):

$$S_{\text{type II,NS-NS}} \sim \frac{1}{\kappa^2} \int d^{10}x \sqrt{-\|G\|}\; e^{-2\Phi} \left(R + |d\Phi|^2 - |dB|^2 \right)$$

We have restricted to the NS-NS sector, which is common to type IIA and IIB.

As in any theory of gravity, the action is highly nonlinear. Note that every term has precisely two spacetime derivatives.

A constant κ has been introduced, with dimensions of length^2. However, because of the factor $e^{-2\Phi}$ in front of the whole Lagrangian, κ is physically irrelevant. We can change it by adding a constant to Φ.

Indeed, if for any reason Φ develops a vacuum expectation value Φ_0, we can scale this out of the action to get a prefactor $e^{-2\Phi_0}$.

It follows that e^{Φ_0} acts like the coupling constant in this theory. We will see independent evidence of this from the worldsheet approach to string theory. Thus we define the string coupling:

$$g_s = e^{\Phi_0}$$

We can also deduce independently from the supergravity action, and from the worldsheet approach, that in general there are derivative corrections.

The supergravity action is unique only if we restrict to terms with two derivatives. Four-derivative terms, for example:

$$R^2, \quad R_{\mu\nu}R^{\mu\nu}, \quad \partial_\mu\partial_\nu\Phi\partial^\mu\partial^\nu\Phi$$

can appear in this action, along with appropriate fermion couplings and couplings to the RR sector. In general, all orders of derivatives are allowed by the symmetries.

However, such terms will require a dimensional constant to appear along with the derivatives. In string theory, this constant turns out to be α', the inverse string tension.

Thus the low energy action for superstring theory can contain terms like

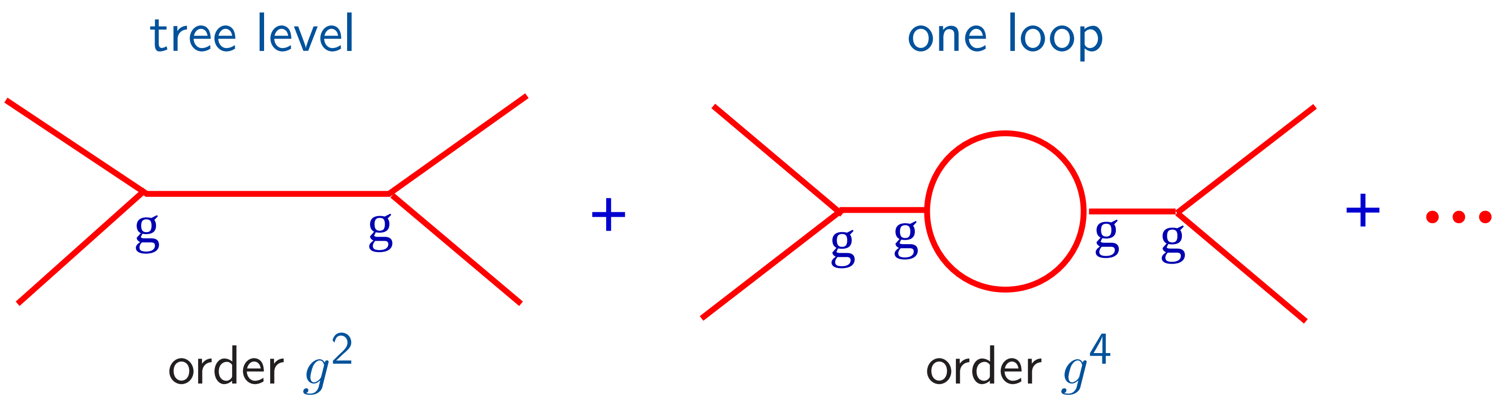
$$\frac{1}{\kappa^2}\int d^{10}x\sqrt{-\|G\|}\; e^{-2\Phi}\left(R+\alpha'(c_1R^2+c_2\,R_{\mu\nu}R^{\mu\nu}+\cdots)+\mathcal{O}(\alpha'^2)\right)$$

We will now discuss superstring amplitudes and confirm all these general arguments.

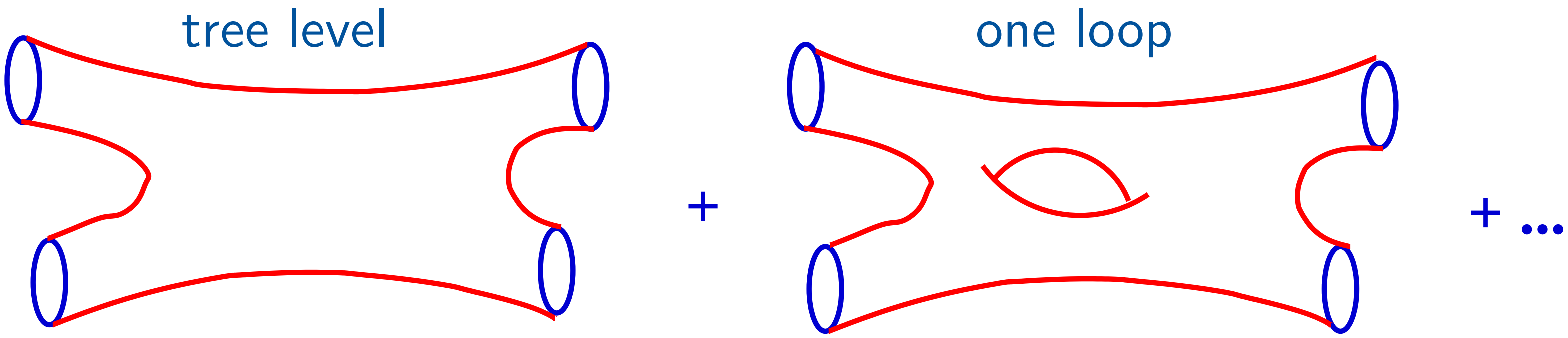
(ii) Superstring interactions

Having quantised free superstrings and obtained their spectrum and symmetries, it is natural to ask how to introduce interactions.

For particles in first-quantised language, we introduce interactions by drawing Feynman diagrams:



For strings, we know that the worldline has to be replaced by a worldsheet. So one may guess that string scattering diagrams (for closed strings) look like:

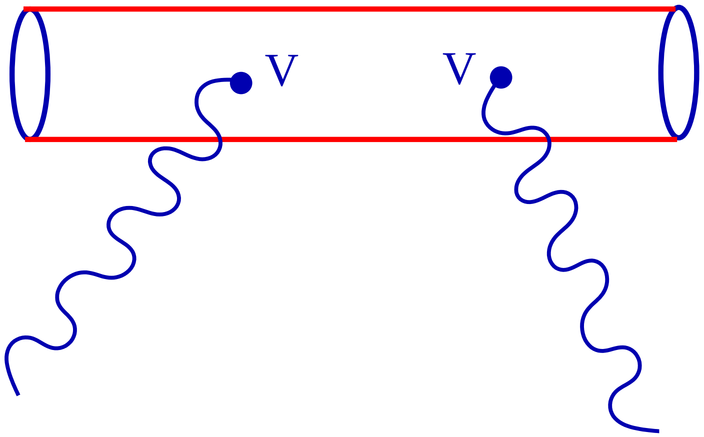


The worldsheet in the tree level diagram is topologically a sphere (with four discs cut out). The worldsheet in the one-loop diagram is a torus, and so on.

We see that the string loop expansion is related to the set of compact two-dimensional manifolds with discs cut out.

How do we quantise strings with worldsheets on such complicated worldsheets? A standard mode expansion cannot be made when there are “handles” on the worldsheet.

For tree-level amplitudes, there is no problem. We start with a cylindrical worldsheet and insert special operators V called vertex operators, to represent absorption and emission of additional strings:



For example, this 4-point amplitude, for open strings, is written as:

$$\langle 1|V_2(k_2)\,\Delta\,V_3(k_3)|4\rangle$$

where Δ is the propagator:

$$\Delta = \frac{1}{L_0} = \int_0^\infty d\tau\, e^{-\tau\, L_0}$$

and

$$L_0 = p^- = \frac{1}{2p^+}\left(\frac{1}{\alpha'}\sum_{m=1}^{\infty}\alpha^i_{-m}\,\alpha^i_m + p^i p^i - a\right)$$

is the worldsheet Hamiltonian.

For loop corrections, things are not so simple. There are two ways to proceed:

(i) make worldsheets of complicated topology by gluing cylinders together. On each cylinder the mode expansion is well-defined. To glue the cylinders together we must define a three string vertex.

(ii) give up the operator formalism we have been using so far, and work with the functional integral. In this case we can define the worldsheet to have any topology we like.

The latter formalism is extremely powerful and allows us to compute string amplitudes quite effectively. In this formalism, every string state has a vertex operator V associated to it, and string amplitudes are simply the correlation function of vertex operators on the given worldsheet.

Whichever way we do things, it turns out that in each formalism, every physical state of the string has a unique vertex operator associated to it.

The computation of the amplitude is completely specified in terms of these operators. As a result, string amplitudes are unique and cannot be postulated by hand.

Thus, for example, the graviton of the closed type II superstring has an operator

$$V_{\mathsf{G}}^{\mu\nu}(k)$$

associated to it, such that the N-graviton scattering amplitude is given symbolically by:

$$\langle V_{\mathsf{G}}^{\mu_1\nu_1}(k_1)\, V_{\mathsf{G}}^{\mu_2\nu_2}(k_2) \cdots V_{\mathsf{G}}^{\mu_{\mathsf{N}}\nu_{\mathsf{N}}}(k_{\mathsf{N}})\rangle$$

The actual computation of string amplitudes is a long story, so we will simply quote the answers and examine their physical properties.

(iii) Closed superstring tree amplitudes

Tree amplitudes are moderately simple for small values of N. For $N = 3$, we find:

$$\langle V_{\mathsf{G}}^{\mu_1\nu_1}(k_1)\, V_{\mathsf{G}}^{\mu_2\nu_2}(k_2)\, V_{\mathsf{G}}^{\mu_3\nu_3}(k_3)\rangle \sim \delta^{10}(k_1 + k_2 + k_3)\; A^{\mu_1\mu_2\mu_3}\, A^{\nu_1\nu_2\nu_3}$$

where

$$A^{\mu_1\mu_2\mu_3} = \eta^{\mu_1\mu_2}(k_1 - k_2)^{\mu_3} \;+\; \text{cyclic permutations}$$

We see that the momentum is conserved, as it should be. This is built in to the structure of vertex operators.

We also see that every term in this amplitude has exactly two momenta.

Multiplying by polarisation tensors

$$\zeta^1_{\mu_1\nu_1}\,\zeta^2_{\mu_2\nu_2}\,\zeta^3_{\mu_3\nu_3}$$

and summing over the indices, we get the three-point amplitude. The expression involves a large number of terms.

Let us try to get a feeling for a typical term:

$$k_1^\alpha\, k_2^\beta\, \zeta^1_{\mu\nu}\, \zeta^{2\mu\nu}\, \zeta^3_{\alpha\beta}$$

As we have seen previously, the polarisation tensors should be thought of as wave functions in momentum space for the corresponding field.

In this case the field is the graviton $G_{\mu\nu}$, or more precisely its fluctuation $h_{\mu\nu}$ away from flat spacetime. And the above expression is just the momentum-space version of:

$$\partial^\alpha h_{\mu\nu}\, \partial^\beta h^{\mu\nu}\, h_{\alpha\beta}$$

It turns out that this term, and all the others of its kind, are precisely the terms in the Einstein Lagrangian:

$$\sqrt{-\|G\|}\, R$$

when we make the replacement

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and expand to cubic order in $h_{\mu\nu}$.

Thus by computing a three-point amplitude following the rules of superstring theory, we are able to read off the effective coupling of the graviton, to cubic order.

The fact that it is equal to the cubic term in Einstein's action is nothing short of a miracle, though one that we expected by now.

This miracle continues at every order, but an extra feature also emerges.

Let us consider the four-graviton amplitude. In this case, it is convenient to work with the Mandelstam variables:

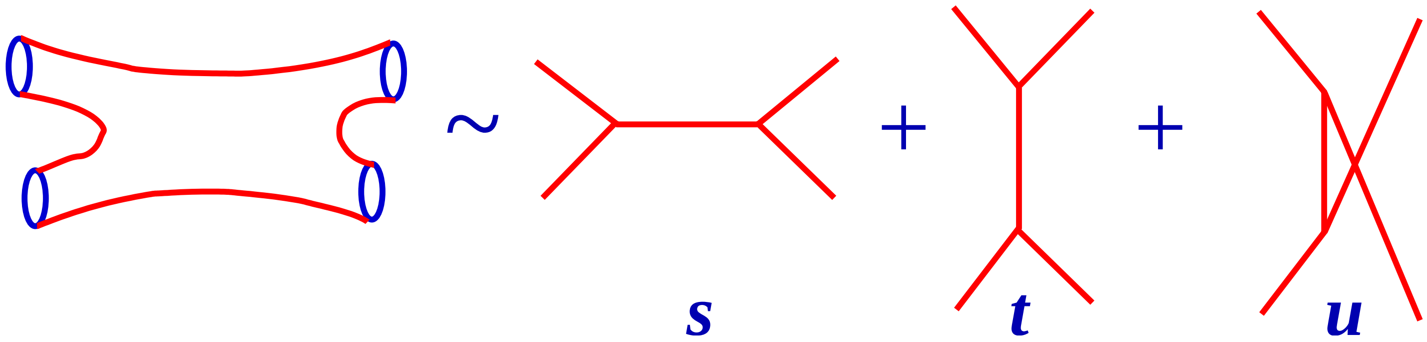
$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 + k_3)^2, \qquad u = -(k_1 + k_4)^2$$

$$\langle V_{\mathsf{G}}^{\mu_1\nu_1}(k_1)\, V_{\mathsf{G}}^{\mu_2\nu_2}(k_2)\, V_{\mathsf{G}}^{\mu_3\nu_3}(k_3)\, V_{\mathsf{G}}^{\mu_4\nu_4}(k_4)\rangle \sim$$
$$\alpha'^3 \, \frac{\Gamma(-\frac{1}{4}\alpha' s)\,\Gamma(-\frac{1}{4}\alpha' t)\,\Gamma_(-\frac{1}{4}\alpha' u)}{\Gamma(1+\frac{1}{4}\alpha' s)\,\Gamma(1+\frac{1}{4}\alpha' t)\,\Gamma(1+\frac{1}{4}\alpha' u)}\; K^{\mu_1\nu_1\cdots\mu_4\nu_4}(\zeta^i, k_i)$$

where $K^{\mu_1\nu_1\cdots\mu_4\nu_4}(k_i)$ is a kinematic factor containing eight powers of momenta.

This is a beautiful expression, with many important properties:

(i) It is completely symmetric in s, t, u. While it corresponds to a single "stringy Feynman diagram", it plays the role of a sum of particle diagrams:



(ii) The amplitude has poles whenever

$$s = \frac{4}{\alpha'}n \quad \text{or} \quad t = \frac{4}{\alpha'}n \quad \text{or} \quad u = \frac{4}{\alpha'}n$$

for any positive integer n.

These are precisely the values of M^2 for which the closed superstring has massive physical states.

The poles occur when any of s, t, u is equal to the mass-squared of a physical state, signalling the possibility of producing these states as resonances in the intermediate channel.

(iii) The amplitude depends on α', so we can expect to see some truly "stringy" properties. Consider the expansion of the gamma-function factor in powers of α':

$$\frac{\Gamma(-\tfrac{1}{4}\alpha' s)\,\Gamma(-\tfrac{1}{4}\alpha' t)\,\Gamma_{(} - \tfrac{1}{4}\alpha' u)}{\Gamma(1+\tfrac{1}{4}\alpha' s)\,\Gamma(1+\tfrac{1}{4}\alpha' t)\,\Gamma(1+\tfrac{1}{4}\alpha' u)} \sim -\frac{64}{\alpha'^3\, s\, t\, u} - 2\zeta(3) + \mathcal{O}(\alpha')$$

The first term cancels 6 momenta in the kinematic factor K, as well as the α'^3 coefficient in the amplitude. Thus it gives a two-derivative term of fourth order in the gravitational fluctuation $h_{\mu\nu}$.

So this must be the fourth order term in the expansion of the Einstein Lagrangian. And indeed it is.

The next term, however, gives a contribution to the effective action which has 8 derivatives and three powers of α. Dimensionally this is consistent, because $\alpha'\partial\partial$ is dimensionless.

In fact, the term in the Lagrangian required to reproduce this term in the amplitude is:

$$\sim \zeta(3)\, \alpha'^3\, t^{\mu_1\nu_1\cdots\mu_4\nu_4}\, t^{\rho_1\sigma_1\cdots\rho_4\sigma_4}\, R_{\mu_1\nu_1\rho_1\sigma_1}\, R_{\mu_2\nu_2\rho_2\sigma_2}\, R_{\mu_3\nu_3\rho_3\sigma_3}\, R_{\mu_4\nu_4\rho_4\sigma_4}$$

where t is some numerical tensor and $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor.

As we predicted, string theory produces not only conventional (Einstein-type) actions with two derivatives, but also higher-derivative corrections to them.

The role of α' is to govern these corrections. It defines what is meant by slowly varying fields. Derivative corrections may be ignored for fields that are slowly varying on the scale of the string length $l_s \sim \sqrt{\alpha'}$.

(iv) Closed superstring loop amplitudes

The loop amplitudes illustrate several key points about superstring theory, which we will now highlight.

(i) Role of the dilaton:

We have alluded before to the role of the dilaton as the coupling constant of string theory.

Let us now see directly how this arises on the worldsheet.

The worldsheet actions we have presented were valid for strings propagating in flat spacetime and in the absence of background fields.

Suppose we allow an arbitrary dilaton field $\Phi(x)$ to be present in spacetime. It turns out that this requires a modification in the worldsheet action:

$$S \to S + \frac{1}{4\pi}\int d\sigma\, dt\, \sqrt{-\|g\|}\, R\, \Phi(X(\sigma, t))$$

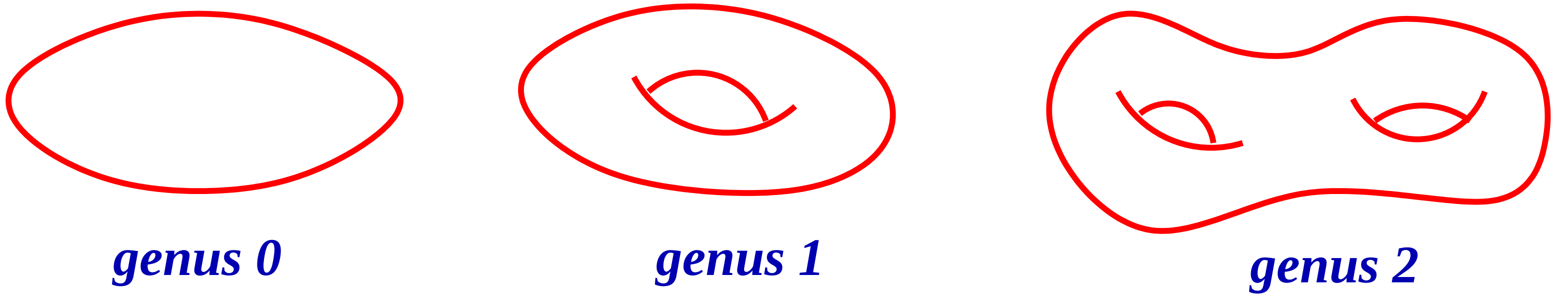
In general, this modification spoils Weyl invariance which was so important in quantising the string. But suppose the dilaton field is constant:

$$\Phi(X) = \Phi_0$$

Then the extra term becomes

$$\frac{\Phi_0}{4\pi} \int d\sigma\, dt \sqrt{-\|g\|}\, R$$

Now, worldsheets with handles are classified by their genus which is just the number of handles:



It is a theorem in differential geometry that on any two-dimensional closed surface,

$$\frac{1}{4\pi}\int d\sigma dt\,\sqrt{-\|g\|}\,R = 2 - 2h$$

where h is the genus.

Thus we see that the modification of the worldsheet action for the constant mode of the dilaton is:

$$S \longrightarrow S + 2\Phi_0(1-h)$$

In the functional integral, we accordingly get the replacement:

$$e^{-S} \longrightarrow e^{-2\Phi_0(1-h)}\,e^{-S} = \left(g_s^2\right)^{h-1} e^{-S}$$

where, as suggested earlier, we have defined:

$$g_s = e^{\Phi_0}$$

We see that every extra handle on the worldsheet comes with an extra power of g_s^2.

In Feynman diagrams each such handle is like a string loop, and we expect each extra loop to be weighted by higher powers of the string coupling.

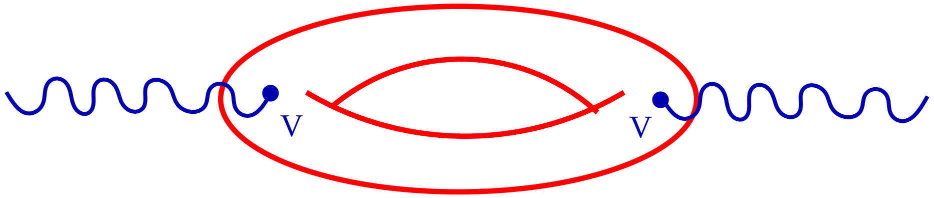
Thus the VEV of the dilaton indeed defines the string coupling constant, confirming the arguments that we earlier advanced on the basis of the spacetime action.

It is a remarkable property of string theory that it has no arbitrary dimensionless constants.

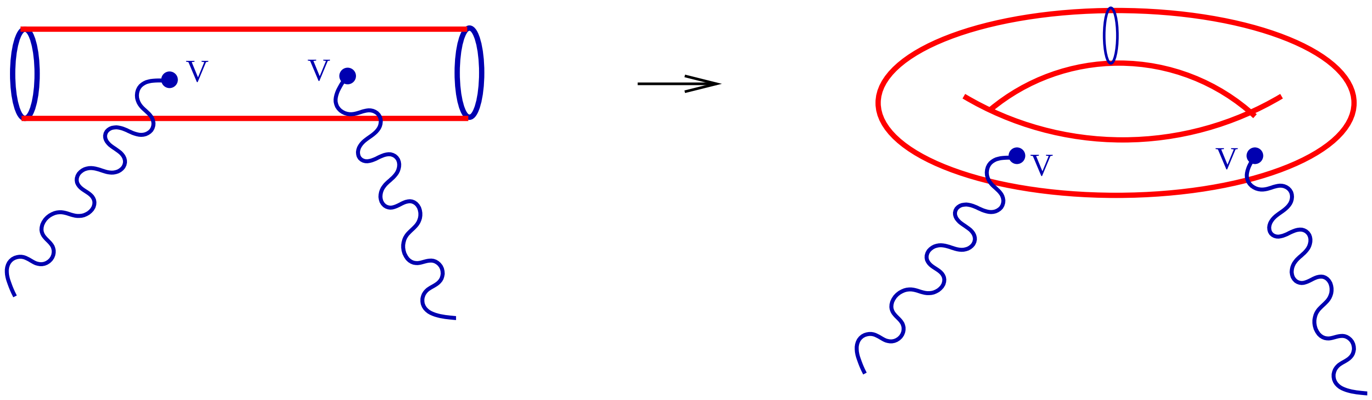
(ii) Finiteness of loop amplitudes:

It is hard to discuss general loop amplitudes without developing more formalism, but one-loop amplitudes are relatively simple.

The reason is that the one-loop diagram:



can be thought of as a cylindrical worldsheet of finite length, rolled up:



The act of rolling up the worldsheet means we are no longer computing a tree amplitude between fixed states, like:

$$\langle 1|V_2(k_2)\,\Delta\,V_3(k_3)|4\rangle$$

but are instead computing a sum like:

$$\sum_n \langle n|\Delta V_2(k_2)\,\Delta\,V_3(k_3)|n\rangle$$

where $|n\rangle$ is a complete set of string states.

This is the same as a trace in Hilbert space:

$$\mathrm{tr}\,\left(\Delta V_2(k_2)\,\Delta\,V_3(k_3)\right)$$

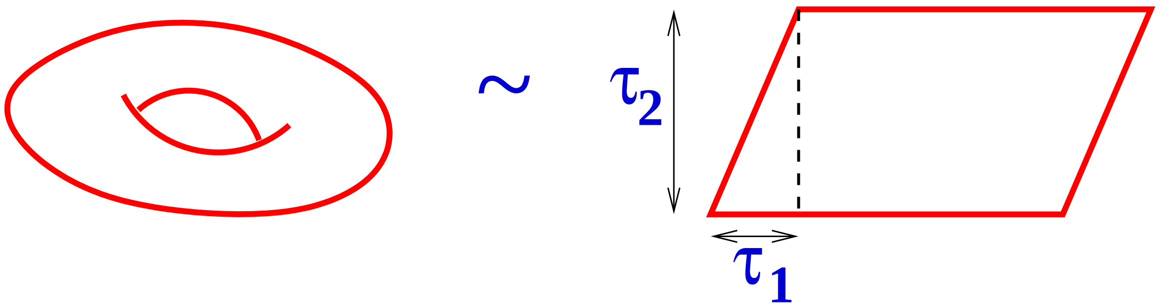
Let us specialise to closed strings. The simplest loop amplitude is the vacuum amplitude, obtained by computing

$$\int \frac{d\tau_1 d\tau_2}{(\tau_2)^2}\ Z(\tau,\overline{\tau})$$

where

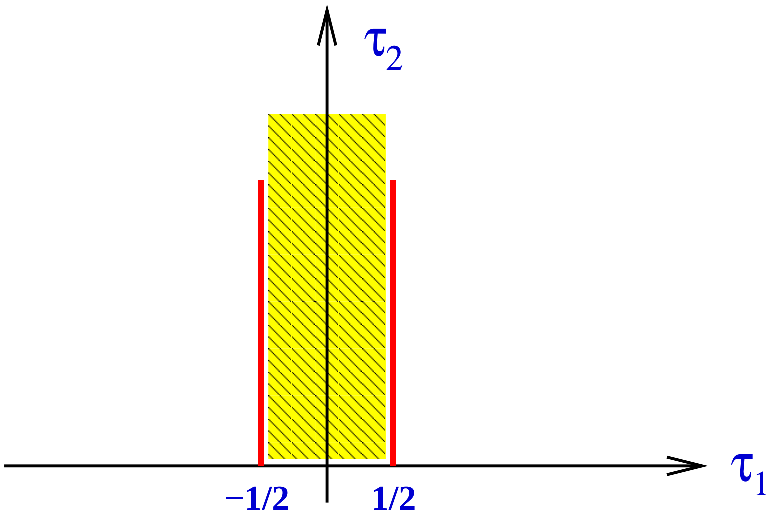
$$Z(\tau,\overline{\tau}) = \mathrm{tr}\ e^{2\pi i\tau L_0} e^{-2\pi i\overline{\tau}\widetilde{L}_0} \sim \mathrm{tr}\ \Delta$$

Here $\tau = \tau_1 + i\tau_2$ describes the shape of the toroidal worldsheet:



$Z(\tau,\overline{\tau})$ is the generating function for the number of states in the string theory, or the partition function.

We must integrate over τ. But what is the range? Naively, it looks like:



Note that the integration over τ_1 enforces the constraint:

$$L_0 = \widetilde{L}_0$$

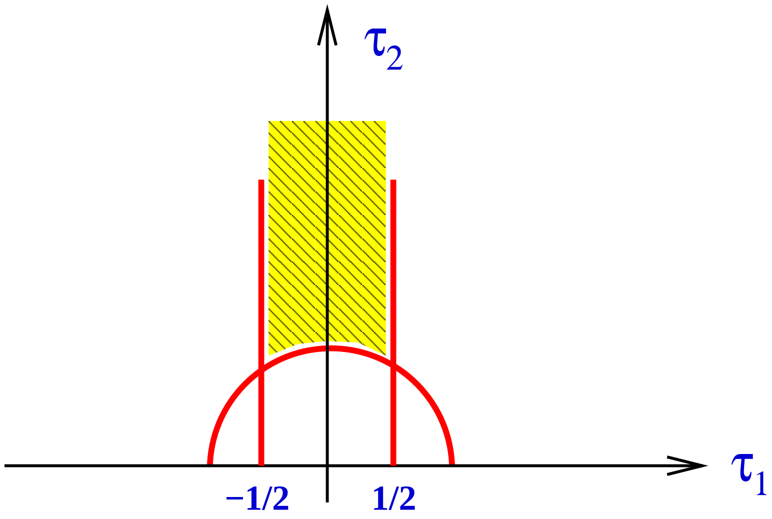
However, the partition function has an infinite discrete symmetry under:

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

where a, b, c, d are integers satisfying $ad - bc = 1$. This symmetry is called modular invariance.

Under this transformation, the shaded region divides into infinitely many equivalent subregions.

Thus to avoid overcounting, we must restrict the integration over τ to a "fundamental region":



Notice that the region near $\tau \sim 0$ is now excluded.

But since $L_0 \sim p^2 + \cdots$, this means the region of large p^2 is excluded.

This cuts off the ultraviolet modes propagating in the loop. As a result, we get a finite answer.

While this example had no external string states, we can repeat the same type of calculation with, for example, external gravitons.

Again, modular invariance ensures that the one-loop amplitude is finite. Indeed, all loop amplitudes have similar invariances and they are all finite.

Thus we have evaded one of the biggest theoretical problems in gravity since the days of Einstein.

Gravity as a field theory is nonrenormalisable and gives unremovable UV divergences. But gravity in string theory has no UV divergences at all!

This is why we believe that string theory is a consistent theory of quantum gravity.